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This volume contains proceedings of the 9th International School-Seminar on Nonlinear Analysis and Extremal Problems (NLA-2026). The 9th International School-Seminar on Nonlinear Analysis and Extremal Problems is a biennial scientific event that takes place in Irkutsk, Russia. NLA-2026 aims at sharing recent advances in various areas of modern nonlinear analysis and exposing young researchers to some fast-paced topics in the field. The school-seminar talks present recent developments in various fields of nonlinear analysis, calculus of variations and control theory, partial differential equations, optimization, dynamical systems, and numerical methods.

This volume is intended for researchers specializing in the corresponding fields of mathematics.

The 9th International School-Seminar on Nonlinear Analysis and Extremal Problems (NLA-2026) is supported by the Mathematical Center in Akademgorodok under the agreement No. 075-15-2025-349 with the Ministry of Science and Higher Education of the Russian Federation.

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Preface

This volume contains proceedings of the 9th International School-Seminar on Nonlinear Analysis and Extremal Problems (NLA-2026). The 9th International School-Seminar on Nonlinear Analysis and Extremal Problems is a biennial scientific event that takes place in Irkutsk, Russia. The school-seminar (NLA-2026) will be held on 22–26 June 2026. NLA-2026 aims at sharing recent advances in various areas of modern nonlinear analysis and exposing young researchers to some fast-paced topics in the field.

The meeting program includes a series of short courses given by the leading experts in the field and talks of researchers.

The 9th International School-Seminar on Nonlinear Analysis and Extremal Problems (NLA-2026) is supported by the Mathematical Center in Akademgorodok under the agreement No. 075-15-2025-349 with the Ministry of Science and Higher Education of the Russian Federation.

The main topics of the school-seminar are

- ❑ Nonlinear analysis and its applications
- ❑ Calculus of variations and Control theory
- ❑ Partial differential equations
- ❑ Optimization
- ❑ Dynamical systems
- ❑ Numerical methods

The school-seminar featured the lecture courses devoted to various aspects of theoretical and applied nonlinear analysis and control theory:

- ❑ *"Fokker-Planck-Kolmogorov Equations: Changes of Coordinates"*
  by **Stanislav Shaposhnikov** (Lomonosov Moscow State University, Russia)
- ❑ *"Minimax and Viscosity Solutions of Path-Dependent Hamilton-Jacobi Equations"*
  by **Mikhail Gomoyunov** (Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, Russia)
- ❑ *"Difference Stabilizing Controller of Linear Delay System"*
  by **Guang-Da Hu** (Department of Mathematics, Shanghai University, Shanghai, China)
- ❑ *"Higher-Order Compact Difference Methods for Nonlinear Dispersive Wave Equations"*
  by **Shuguang Li** (School of Science, Dalian Maritime University, Dalian, China)
- ❑ *"Compatibility of Collocation Error Analyses for Volterra Integral Equations with Smooth and Weakly Singular Kernels"*
  by **Hui Liang** (School of Science, Harbin Institute of Technology, Shenzhen, China)

Organization

The 9th International School-Seminar on Nonlinear Analysis and Extremal Problems is organized by Matrosov Institute for System Dynamics and Control Theory of SB RAS in cooperation with Novosibirsk State University, Krasovskii Institute of Mathematics and Mechanics of UB RAS, Irkutsk State University, and Mathematical Center in Akademgorodok (Novosibirsk).

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Contents

Observation Problem on Part of Coordinates under Integral Constraints*

Boris Ananyev

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Observation problem for a linear system is considered, part of coordinates of which is accessible. In a sense the solution of such problem is more difficult of earlier considered even in a quadratic case. Disturbances in a system are assumed to belong to the spaces L_p, $1 \le p \le \infty$.

Keywords: observability, integral constraints, convex analysis

Consider a linear non-stationary system

$$\dot{x} = A(t)x + B(t)v(t), \quad t \in [0,T], \quad \|v(\cdot)\|_p \le 1, \tag{1}$$

where $A(t) \in \mathbb{R}^{(m+n)\times(m+n)}$, $B(t) \in \mathbb{R}^{(m+n)\times r}$, $\|v(\cdot)\|_p = \left(\int_0^T |v(t)|_p^p \, dt\right)^{\frac{1}{p}}$ is the norm in Banach space $L_p^r[0,T]$, $1 < p \le \infty$. Hereinafter, $|v|_p = \left(\sum_{i\in 1:r} |v_i|^p\right)^{\frac{1}{p}}$ is the p-norm for vectors $v \in \mathbb{R}^r$. In case $p = \infty$ we set $\|v(\cdot)\|_\infty = ess\,sup_{t\in[0,T]}|v(t)|_\infty$ and $|v|_\infty = \max\{|v_1|, \dots, |v_r|\}$ is the ∞-norm. Elements of matrices $A(t)$, $B(t)$ belong to the space $L_\infty[0,T]$. The function $v(\cdot)$ is considered as a perturbation.

Let a solution $x(t) = [x_1(t); x_2(t)]$ of system (1) consists of m first observable coordinates $x_1(t)$ and n last non-observable coordinates $x_2(t)$. So, the signal has a view

$$y(t) = x_1(t) = Gx(t), \quad G = [I_m \;\; O_{m\times n}], \tag{2}$$

where I_m is unit $m \times m$-matrix, $O_{m\times n}$ is zero $m \times n$-matrix. **The problem considered here** consists in estimation of the vector $x_2(T)$ according to observation $x_1(\cdot)$. In [1,2], the method of the solution of such problems was offered. It based on creation in phase space of *information set* (further IS) of various vectors $x_2(T)$ compatible to measurements and restrictions for perturbations. It is necessary to notice that mostly there is an additional disturbance $w(\cdot)$ in observation $y(t) = x_1(t) + w(t)$ and this function also restricted.

However, the vector of $y(t)$ allows differentiation

$$z(t) = \dot{y}(t) = GA(t)x(t) + GB(t)v(t), \tag{3}$$

where the equality is understood almost everywhere. If the differentiation is possible, we give the definition: **IS** $\mathcal{X}_T(y,z) \subset \mathbb{R}^n$ **consists of vectors** $x_2(T)$, for which there is a function $v(\cdot)$ with $\|v(\cdot)\|_p \le 1$ and equalities (2), (3) are fulfilled, where $x(t)$ is the solution of (1) with boundary condition $x(T) = [y(T); x_2(T)]$. Let $\tilde{G} = [O_{n\times m} \;\; I_n]$. Using minimax theorem [3] and results of convex analysis [4], we obtain the following.

* The work was performed as part of research conducted in the Ural Mathematical Center with the financial support of the Ministry of Science and Higher Education of the Russian Federation (Agreement number № 075-02-2026-737).

Theorem 1. *IS $\mathcal{X}_T(y,z)$ is closed and convex. Its support function is defined by the formula*

$$\rho(l \mid \mathcal{X}_T(y,z)) = \begin{cases} \inf\limits_{\lambda(\cdot),\mu(\cdot)} \Big\{ \displaystyle\int_0^T (\lambda'(t)y(t) + \mu'(t)z(t))\,dt - s_1'(T)y(T) & \\ \qquad + \|B'(t)(s(\cdot) - G\mu(\cdot))\|_q \;:\; s_2(T) = l \Big\}, & \text{if } l \in \min \mathbb{N}_T; \\ +\infty, & \text{if } l \notin \min \mathbb{N}_T. \end{cases} \tag{4}$$

The set $\mathcal{X}_T(y,z)$ is bounded if and only if $\det \mathbb{N}_T \neq 0$. Here $'$ means the transposition; $q = p/(p-1)$; $\lambda(\cdot) \in L_2^m[0,T]$, $\mu(\cdot) \in L_q^m[0,T]$; the observability matrix

$$\mathbb{N}_T = \int_0^T \tilde{G}X'(u,T)\,(A'(u)G'GA(u) + G'G)\,X(u,T)\tilde{G}'\,du.$$

The function $s(\cdot) = [s_1(\cdot); s_2(\cdot)]$ is a solution of the conjugate system

$$\dot{s} = -A'(t)(s - G'\mu(t)) + G'\lambda(t), \quad s(0) = 0.$$

If differentiation of the signal is impossible, then the multiplier $\mu(\cdot) = 0$ in (4). In that case, we have IS $\mathbf{X}_T(y) \supset \mathcal{X}_T(y,z)$ and the observability matrix has the form

$$\mathbf{N}_T = \int_0^T \tilde{G}X'(u,T)G'GX(u,T)\tilde{G}'\,du.$$

Besides, in this case the parameter p may be equal 1. In this presentation, we discuss the relation between sets $\mathbf{X}_T(y)$ and $\mathcal{X}_T(y,z)$. In case without differentiation, the approximation scheme with a signal $y(t) = Gx(t) + \varepsilon^2 w(t)$ is considered, where ε is a small parameter and $w(\cdot)$ is an additional disturbance, for which the constraint $\|v(\cdot)\|_p^p + \|w(\cdot)\|_2^2 \leq 1$ is valid. Using this approximation, we obtain the set

$$\mathbf{X}_T^\varepsilon(y) = \{x_2 : V(T, [x_1(T); x_2]) \leq 1\}, \tag{5}$$

where the function $V(t,x)$ is a solution of partial differential equation $(1 < p < \infty)$

$$V_t = |V_x B(t)|_q^q \left(1/p^q - 1/p^{q/p}\right) - V_x A(t)x + |y(t) - Gx|_2^2/\varepsilon^2, \quad V(0,x) = 0. \tag{6}$$

The inclusion $\mathbf{X}_T(y) \subset \mathbf{X}_T^\varepsilon(y)$ is valid. For $p = 2$, this equation is solved exactly: $V(t,x) = x'P(t)x - 2d'(t)x + e(t)$. The parameters satisfy the singularly perturbed differential equation.

Example 1. For the system $\dot{x}_1 = v$, $\dot{x}_2 = x_1$, $y(t) = x_1(t)$, we have $m = n = 1$ and $\mathbb{N}_T = \mathbf{N}_T = 0$. Therefore, $\mathcal{X}_T(y,z) = \mathbf{X}_T(y) = \mathbb{R}$ and the second coordinate cannot be found.

Example 2. For the system $\dot{x}_1 = x_2$, $\dot{x}_2 = v$, $y(t) = x_1(t)$, we have $m = n = 1$ and $\mathbf{N}_T > 0$. IS $\mathcal{X}_T(y,z) = \{z(T)\}$ is a singleton, but $\mathbf{X}_T(y)$ is a segment, whose length depends on number p. For $p = 2$, it can be proved with the help of (5), (6) that the length of the segment is very small but not equals zero.

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About One Method for Solving a Partially Boolean Programming Problem Based on Gradient Descent and the Adaptive Method of Centers*

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This paper examines a sequential unconstrained minimization method based on the adaptive center method and gradient descent strategies. The proposed technique is designed for partially Boolean linear programming tasks. Its effectiveness is demonstrated through a practical case study: solving a non-guillotine cutting and packing problem, where a set of rectangular pieces must be optimally placed on a half-strip.

Keywords: partially Boolean linear programming task, method of centers, gradient descent method

Practical optimization problems often have properties that prevent the use of common optimization methods for solving them. Partially Boolean linear programming problems belong to this class of tasks. The Boolean nature of certain variables in the problem prevents the application of standard linear programming methods (such as the simplex method) to solve them or requires significantly complicating the procedure – for example, by repeatedly solving simplified subproblems that can be solved by the simplex method.

Among such problems is the non-guillotine packing of rectangular details on a half-strip [1]. The problem is defined as follows:

$$A \to min$$
$$x_i + (b_i - a_i)z_i \le A - a_i \; i = 1 \dots n$$
$$y_i + (a_i - b_i)z_i \le B - b_i \; i = 1 \dots n$$
$$-x_i + x_j - (b_i - a_i)z_i + \bar{A}t_{ij} + \bar{A}s_{ij} \ge a_i \; i = 1 \dots n-1, j = i+1 \dots n$$
$$x_i - x_j - (b_j - a_j)z_j + \bar{A}t_{ij} - \bar{A}s_{ij} \ge a_j - \bar{A} \; i = 1 \dots n-1, j = i+1 \dots n$$
$$-y_i + y_j - (a_i - b_i)z_i - \bar{B}t_{ij} + \bar{B}s_{ij} \ge b_i - \bar{B} \; i = 1 \dots n-1, j = i+1 \dots n$$
$$y_i - y_j - (a_j - b_j)z_j - \bar{B}t_{ij} - \bar{B}s_{ij} \ge b_j - 2\bar{B} \; i = 1 \dots n-1, j = i+1 \dots n$$
$$x_i \ge 0 \; i = 1 \dots n$$

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$$y_i \geq 0 \; i = 1 \dots n$$

$$z_i \in \{0, 1\} \; i = 1 \dots n$$

$$s_{ij} \in \{0, 1\} \; i = 1 \dots n-1, j = i+1 \dots n$$

$$t_{ij} \in \{0, 1\} \; i = 1 \dots n-1, j = i+1 \dots n$$

where x_i, y_i – are the coordinates of the placement of the i-th rectangle (size of $a_i \times b_i$); z_i determines the horizontal or vertical orientation of the detail; the set of Boolean variables s_{ij} and t_{ij} is used to define the relative positioning of each pair of details with respect to each other and to ensure they do not overlap; the variable A presents the unknown length of the half-strip required to accommodate the set of rectangles. The constants $\bar{A}$ and $\bar{B}$ are chosen to be no smaller than the dimensions of a sheet sufficient to fit the entire set of rectangles.

The paper proposes a computational procedure based on a combination of gradient descent methods and an adaptive version of one of the sequential unconstrained optimization methods, the method of centers, for solving constrained optimization problems.

When applying the method of centers, the inequalities from the constraint system (except for the Boolean variable constraints) are represented in the form $g_j(x) \leq 0$, and then a convolution of the constraint functions is performed: $g(x) = \max\{g_j(x), j = 1 \dots m\}$, where m is the number of constraints in the optimization problem. The method of centers, as a sequential unconstrained optimization technique, replaces the solution of the original problem with a series of unconstrained problems having the objective function $F_k(x) = \max\{f(x) - f(x_k) + \varepsilon, g(x)\}$, where $f(x)$ is the objective function and x_k is a current iteration point; $\varepsilon > 0$ is an adaptation parameter that, under certain conditions, guaranties a decrease in the objective function by at least ε. The next function in the center method is constructed upon reaching a feasible point y at which the following holds: $f(x) < f(x_k) - \varepsilon$. In this case, it is assumed that $x_{k+1} = y$.

The initial iteration point is determined by any feasible set of variable values. In the problem of obtaining a non-guillotine arrangement of a set of rectangles on a half-strip, the first feasible solution can be easily obtained by placing all pieces side by side horizontally. Accordingly, the variable A is defined as the sum of the lengths of all pieces.

The minimization of the function $F_k(x)$ is carried out using a procedure similar to the coordinate descent algorithm. Due to the non-differentiability of the function $F_k(x)$, the subgradient of the function is used instead of the gradient; in particular, any convex combination of the gradients of the active component functions of $F_k(x)$ can be used.

The iterative process within the modified coordinate descent procedure is performed by selecting, based on the subgradient at the current iteration point, the coordinate of a Boolean variable whose value change to the opposite one may lead to a decrease in the objective function. If the subgradient with respect to the Boolean variables has zero values, a standard gradient descent algorithm can be applied, adapting the non-Boolean coordinates of the desired variable vector.

The computational experiment conducted for the problem of obtaining a non-guillotine placement of a set of details on a half-strip demonstrates the fundamental applicability of this procedure. Another advantage of the algorithm is that a feasible arrangement is available at every iteration of the process. This allows one to stop the computational procedure at any moment and still have a feasible solution to the problem.

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Direct and Inverse Problems of HIV Dynamics

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The report is devoted to applying the dynamic regularization method to investigating mathematical HIV model and restoring a stabilizing control.

Keywords: regularization, HIV, control

The report describes application of the dynamic regularization method [1] to restoring control acting on a system of ordinary differential equations. The dynamic regularization method consists of consecutive construction on the sequence of time intervals of an approximate trajectory $z_h(t)$ of a special system of ordinary differential equations, calling a ***leader***, and with its help — of further restoration of the control. At present the convergence of the dynamic regularization method is proved for a class of systems of ordinary differential equations satisfying the Lipschitz condition in the statement of this method, imposing generally accepted but rather strict conditions on the class of problems to be solved. In this case the convergence is understanding as the uniform approaching of the leader to the trajectory of a system under consideration, and the restoring control in the metric of the space L_2 to the so-called ***normal*** control, that has the minimal L_2-norm, under the condition of decreasing parameters of applying regularization method.

We consider a classic mathematical HIV model [2] which includes an external medical effects. The mathematical model under consideration has the general form

$$\dot{x}(t) = G(x(t)) + F \cdot u(t),\ t \in [0, T], \tag{1}$$

where $[0, T]$ is the entire time interval, $u(t)$ is the desired control, F is a constant vector and G is a nonlinear function. It is important to note that because of non-linearity the system (1) does not satisfy the Lipschitz condition required in dynamic regularization method, however the presented in the report approach allows one to consider the approximation of the system (1) which satisfies the Lipschitz condition and utilizing techniques developed by the authors to obtain a good approximating results.

Theoretical reasoning supported by practical application shows that the best approximation results are archived when the system is stabilizing which means that the measuring data is asymptotically constant. For the system

$$\dot{x}(t) = G(x(t)),\ t \in [0, T], \tag{2}$$

its *steady* states $\{\overline{x}\}$, to which the solutions tend, are found in [2]. This allows us to linearize the system (1) in a neighborhood of one of the steady states of the system (2) in order to apply the dynamic regularization method with the Lipschitz condition. It should be noted that a single linear approximation is not satisfactory for the entire time interval, so we propose to linearize the system with a given step, as a matter of fact decomposing the original system into a Taylor series of the first order with respect to restoring values of the leader. Step-by-step reconstruction of trajectory and control by the dynamic regularization method allows one effectively use such expansion in distinction to static methods which restore required functions on the whole interval as a single step. Finally at every next set of time intervals it is necessary to apply the dynamic regularization method to the following linearized system

$$\dot{x}(t) = G(x(t)) + F \cdot u(t) \approx G(z_h(\xi_j)) + \frac{\partial G(\overline{x})}{\partial x}(x(t) - z_h(\xi_j)) + F \cdot u(t), \tag{3}$$

$$t \in [\xi_j, \xi_{j+1}] \subseteq [0, T].$$

Below on the figure 1 one can see a control, acting on the system (1), restored by the described above method by the linearized system (3).

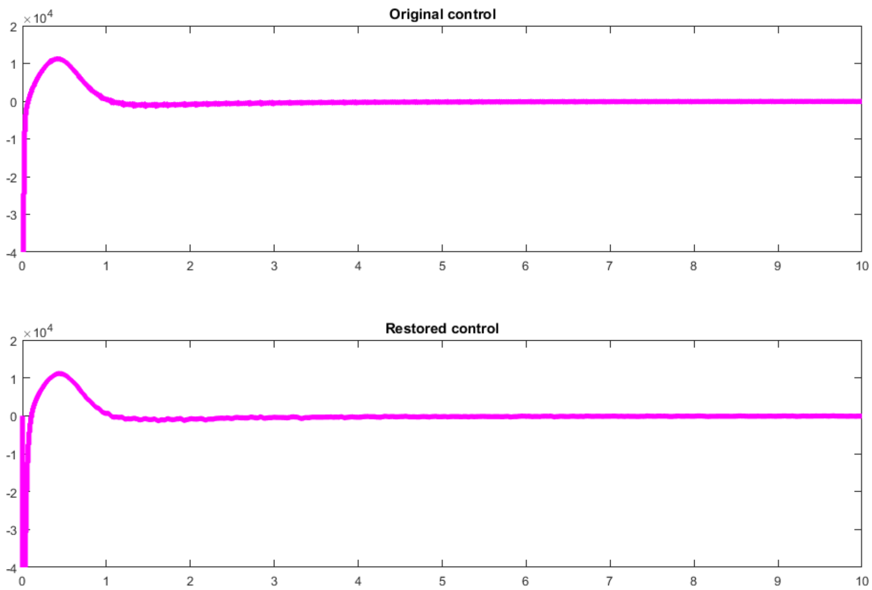

Figure 1: Example of original and restored controls

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Best Approximation by Radial Basis Function Networks with Two Fixed Centers

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We study the approximation of continuous multivariate functions by radial basis function neural networks with two fixed centers, in which all units share a common smoothing factor and incorporate additional translations. We establish a formula for the exact computation of the approximation error in the uniform norm and obtain a characterization of a best approximation.

Keywords: RBFNN, path, extremal path, best approximation, approximation error

Radial basis functions (RBFs) form a family of multivariate functions whose values depend only on the distance from a prescribed center. In other words, for a center $\mathbf{c}$ and a radius ρ, an RBF takes the same value at all points $\mathbf{x}$ satisfying $\|\mathbf{x}-\mathbf{c}\| = \rho$. Based on RBFs, Broomhead and Lowe [3] introduced *radial basis function neural networks* (RBFNNs), which have proved to be effective in various applications.

Let $f(\mathbf{x}) = f(x_1, \dots, x_d)$ be a continuous function defined on a compact set $Q \subset \mathbb{R}^d$. We approximate f by elements of the following class of RBFNNs with a fixed activation function $g : \mathbb{R} \to \mathbb{R}$, a fixed smoothing factor σ, fixed centers $\mathbf{c}_1, \mathbf{c}_2$, and with varying translations θ_i

$$\mathcal{S} = \mathcal{S}(g, \mathbf{c}_1, \mathbf{c}_2, \sigma) = \left\{ \sum_{i=1}^{m} w_i\, g\left(\frac{\|\mathbf{x}-\mathbf{c}_i\|}{\sigma} - \theta_i \right) :\ w_i, \theta_i \in \mathbb{R},\ \mathbf{c}_i \in \{\mathbf{c}_1, \mathbf{c}_2\} \right\}.$$

The error of approximation is defined by

$$E(f) = E(f, \mathcal{S}) = \inf_{h \in \mathcal{S}} \|f - h\|, \qquad \|f-h\| = \max_{\mathbf{x} \in Q} |f(\mathbf{x}) - h(\mathbf{x})|.$$

An element $u \in \mathcal{S}$ is called a best approximation to f from $\mathcal{S}$ if

$$\|f - u\| = E(f, \mathcal{S}).$$

Definition 1. (see [1]) *A finite or infinite ordered set $p = (\mathbf{p}_1, \mathbf{p}_2, \dots) \subset \mathbb{R}^d$, with $\mathbf{p}_i \neq \mathbf{p}_{i+1}$, is called a* path *with respect to the centers $\mathbf{c}_1$ and $\mathbf{c}_2$ if it satisfies one of the following conditions:*

(i) $\|\mathbf{p}_1 - \mathbf{c}_1\| = \|\mathbf{p}_2 - \mathbf{c}_1\|$, $\|\mathbf{p}_2 - \mathbf{c}_2\| = \|\mathbf{p}_3 - \mathbf{c}_2\|$, $\|\mathbf{p}_3 - \mathbf{c}_1\| = \|\mathbf{p}_4 - \mathbf{c}_1\|$, *and so on, alternating between the centers.*

(ii) $\|\mathbf{p}_1 - \mathbf{c}_2\| = \|\mathbf{p}_2 - \mathbf{c}_2\|$, $\|\mathbf{p}_2 - \mathbf{c}_1\| = \|\mathbf{p}_3 - \mathbf{c}_1\|$, $\|\mathbf{p}_3 - \mathbf{c}_2\| = \|\mathbf{p}_4 - \mathbf{c}_2\|$, *and so on, alternating in the opposite order.*

Definition 2. *A finite or infinite path $(\mathbf{p}_1, \mathbf{p}_2, \cdots)$ is said extremal for a function $f \in C(Q)$ if it satisfies one of the following conditions:*

(1) $f(\mathbf{p}_i) = (-1)^i \|f\|, i = 1, 2, \cdots,$ or

(2) $f(\mathbf{p}_i) = (-1)^{i+1}\|f\|, i = 1, 2, \cdots$.

Let us also consider the following class of radial functions

$$\mathcal{D} = \{r_1(\|\mathbf{x}-\mathbf{c}_1\|) + r_2(\|\mathbf{x}-\mathbf{c}_2\|) : \ r_i \in C(\mathbb{R}), \ i = 1, 2\}.$$

In the following, we use the proximinality of $\mathcal{D}$. The set $\mathcal{D}$ is said to be proximinal in $C(Q)$ if for every $f \in C(Q)$ there exists $v \in \mathcal{D}$ such that

$$\|f - v\| = \inf_{h \in \mathcal{D}} \|f - h\|.$$

The images of the distance functions $\|\mathbf{x}-\mathbf{c}_1\|$ and $\|\mathbf{x}-\mathbf{c}_2\|$ on the compact set Q are denoted by X_1 and X_2, respectively. For any function $h \in C(Q)$, let us define the following real-valued functions:

$$s_1(a) = \max_{\substack{\mathbf{x}\in Q \\ \|\mathbf{x}-\mathbf{c}_1\|=a}} h(x), \ s_2(a) = \min_{\substack{\mathbf{x}\in Q \\ \|\mathbf{x}-\mathbf{c}_1\|=a}} h(x), \ a \in X_1, \tag{1}$$

$$w_1(b) = \max_{\substack{\mathbf{x}\in Q \\ \|\mathbf{x}-\mathbf{c}_2\|=b}} h(x), \ w_2(b) = \min_{\substack{\mathbf{x}\in Q \\ \|\mathbf{x}-\mathbf{c}_2\|=b}} h(x), \ b \in X_2. \tag{2}$$

Continuity of these functions is studied in [2].

Theorem 1. *Let $Q \subset \mathbb{R}^d$ be a compact set such that $\mathcal{D}$ is proximinal in $C(Q)$. Let the activation function $g \in C(\mathbb{R})$ be bounded, nonconstant, and have a finite limit at $+\infty$ or $-\infty$. Assume further that the functions* (1) *and* (2) *are continuous. Then, for any $f \in C(Q)$, the approximation error of the RBFNN class $\mathcal{S}(g, \mathbf{c}_1, \mathbf{c}_2, \sigma)$ can be computed by the formula*

$$E(f, \mathcal{S}) = \sup_{p \subset Q} |G_p(f)|,$$

where the supremum is taken over all closed paths in Q.

Theorem 2. *Assume the hypotheses of Theorem 1 hold. Let $f \in C(Q)$ and let $u \in \mathcal{S}$. Then u is a best approximation to f from $\mathcal{S}$ if and only if one of the following two conditions holds:*

1. *There exists a closed path $p \subset Q$ that is extremal for $f - u$.*
2. *For every $N \in \mathbb{N}$ there exists a (not necessarily closed) path $p \subset Q$ consisting of N points that is extremal for $f - u$.*

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Tomographic Reconstruction of Vortex Vector Fields Using Cylindrical Vector Wave Functions*

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This work addresses the inverse problem of reconstructing a three-dimensional solenoidal vector field $G(r)$ from its longitudinal ray transforms. We employ a specialized solenoidal basis derived from Bessel beams. The method is validated on a synthetic solenoidal field representing vortex structures relevant to plasma physics and fluid dynamics and represents Solov'ev solution for the Grad-Shafranov equation for plasma equilibrium. Longitudinal ray transform measurements are computed along rays with varying orientations, and expansion coefficients in the cylindrical vector wave functions basis are recovered via truncated SVD. The method demonstrates robust reconstruction capabilities for vector fields with global topological structure (vortices) that cannot be adequately captured by point-wise measurements alone.

Keywords: inverse problems, Harmonic analysis, Radon transform, Bessel functions, Cylindrical vector wave functions

1 Introduction

Tomographic reconstruction of vector fields presents unique challenges compared to scalar fields due to directional sensitivity and potential singularities. Traditional approaches based on Fourier or Cartesian bases often fail to capture global topological features such as vortices or magnetic flux tubes. Cylindrical vector wave functions provide a natural basis for vortex vector fields with singular behavior on the symmetry axis, commonly encountered in plasma physics, fluid dynamics, and atmospheric science. This work investigates the reconstruction of a synthetic solenoidal field G with $1/\rho$ singularity, representing an idealized vortex filament. We present tomographic reconstruction from longitudinal ray transform measurements.

2 The reconstructuin method

To satisfy the physical constraint $\nabla \cdot G = 0$, the field is expanded into a set of solenoidal eigenfunctions (Hansen cylindrical vector functions). The field is decomposed into toroidal (M) and poloidal (N) components:

$$G(r) = \sum_{n,k} A_{n,k} M_{n,k}(r) + B_{n,k} N_{n,k}(r). \tag{1}$$

The resulting field G features a characteristic poloidal vortex and a toroidal component G_ϕ, providing a rigorous benchmark for the reconstruction of complex magnetic topologies.

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2.1 Basis functions

Cylindrical vector wave functions basis consists of two families:

$$M_{nk} = \nabla \times \left[e_z J_n(k_\rho \rho) e^{in\phi} e^{ik_z z}\right], \quad N_{nk} = \frac{1}{k}\nabla \times M_{nk},$$

where J_n are Bessel functions of the first kind. For numerical stability near $\rho = 0$, we employ recurrence relations to avoid explicit division by ρ.

2.2 Longitudinal ray transform

The longitudinal ray transform (LRT) is defined as:

$$\mathcal{J}G(u,\xi) = \int_{-\infty}^{\infty} G(u + t\xi)\cdot \xi \, dt, \tag{2}$$

where $\xi = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$ defines ray orientation.
By linearity of the Ray transform, the measured data can be written as

$$\mathcal{J}G(u,\xi) = \sum_{n,k} A_{n,k}\mathcal{J}M_{n,k}(u,\xi) + B_{n,k}\mathcal{J}N_{n,k}(u,\xi). \tag{3}$$

The ray transforms of M_{nk} and N_{nk} are evaluated analytically.

$$\mathcal{J}M_{n,k}(u,\xi) = i(-i)^n \sum_{\sigma=\pm 1} \sigma\, e^{in(\alpha+\sigma\phi_0)}\, e^{i\Phi_\sigma}\, \Theta\!\left(\frac{k_\perp}{k} - |\cos\beta|\right), \tag{4}$$

$$\mathcal{J}N_{n,k}(u,\xi) = (-i)^n \frac{k\cos\beta}{D(\beta)} \sum_{\sigma=\pm 1} e^{in(\alpha+\sigma\phi_0)}\, e^{i\Phi_\sigma}\, \Theta\!\left(\frac{k_\perp}{k} - |\cos\beta|\right), \tag{5}$$

where

$$D(\beta) = \sqrt{k_\perp^2 - k^2\cos^2\beta}, \quad \phi_0 = \arccos\left(-\frac{k_z\cos\beta}{k_\perp\sin\beta}\right), \quad \Phi_\sigma = \sigma\frac{D(\beta)}{\sin\beta}u_1 - \frac{k_z}{\sin\beta}u_2,$$

and Θ is the Heaviside step function.

2.3 Reconstruction algorithm

The problem is reduced to a linear system $P = KX$, where K is the integral operator matrix. The system is solved using the Singular Value Decomposition (SVD) method. Numerical experiments showed that: 1) The reconstructed spectral coefficients $\{A, B\}$ perfectly match the direct decomposition of the Soloviev model, 2) The effective rank of the system (typically ≈ 32 for the tested configurations) allows for stable reconstruction with minimal radial harmonics ($nk = 15 - 30$), 3) Tomographic approach providing stable coefficient estimation. This demonstrates key advantage of tomography for fields with global topological structure.

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Transmutation Operators Method for the Cauchy Problem of a Loaded Hyperbolic Equation with a Bessel Operator

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This work studies the Cauchy problem for an inhomogeneous hyperbolic equation with a Bessel operator and a fractional loaded term. Using transmutation operators, the problem is reduced to an auxiliary wave equation. An explicit integral representation of the solution is obtained via d'Alembert formulas and Volterra integral equations.

Keywords: Cauchy problem, hyperbolic equation, Bessel operator, loaded term, Volterra integral equation

We consider the Cauchy problem for the following inhomogeneous hyperbolic equation with a Bessel operator and a loaded term

$$u_{tt} + \frac{2\beta}{t} u_t - L(u) = f(x,t) + D_{0t}^{-\alpha} u(x_0, t), \qquad x \in \mathbb{R}^n,\ t > 0 \tag{1}$$

with homogeneous initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \tag{2}$$

where L – arbitrary linear differential operator acting only on variables $x = (x_1, x_2, \dots, x_n)$, $\beta > 0$, $D_{0t}^{-\alpha}$ is the Riemann-Liouville fractional integral operator of order $\alpha > 0$, and $x_0 \geq 0$. To solve problem (1)–(2), we apply the method of transmutation operators [1,3].

In what follows, we restrict ourselves to the one-dimensional case $n = 1$ and take $L \equiv \frac{\partial^2}{\partial x^2}$. By applying Erdélyi–Kober operators in the form

$$u(x,t) = I_{-\frac{1}{2},\beta}^{(t)} v(x,t), \qquad x \in \mathbb{R}, \tag{3}$$

where $v(x,t)$ is an unknown twice continuously differentiable function, the problem is reduced to auxiliary hyperbolic equations of the form

$$v_{tt} - v_{xx} = F(x,t) + G(x_0, t), \quad x \in \mathbb{R},\ t > 0, \tag{4}$$

with the initial conditions

$$v(x,0) = 0, \qquad v_t(x,0) = 0. \tag{5}$$

Here,

$$F(x,t) = \left(I_{-\frac{1}{2},\beta}^{(t)} \right)^{-1} f(x,t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_0^t (t^2 - s^2)^{-\beta} s^{2\beta} f(x,s) ds,$$

and

$$G(x_0, t) = \frac{1}{\Gamma(1-\beta)\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t^2 - s^2)^{-\beta} s^{2\beta} \left(\int_0^s (s-\tau)^{\alpha-1} u(x_0, \tau)\, d\tau \right) ds.$$

The reduced problem (4)–(5) is analyzed by combining d'Alembert-type solution formulas with the theory of Volterra integral equations [2]. This approach makes it possible to treat the loaded term in a natural way and to derive the associated Volterra integral equation for the unknown trace function. The corresponding resolvent kernels are obtained explicitly, which enables the construction of closed integral representations of the solution. As a result, the solution of the original problem is represented in explicit form in terms of Bessel functions and related special functions arising naturally in the transmutation procedure [3].

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Unique Solvability of the Problem for a Loaded Hyperbolic Equation with Variable Coefficients

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This paper establishes the unique solvability of a boundary value problem for a loaded integro-differential equation with a hyperbolic operator.

Keywords: loaded equation, hyperbolic operator, integro-differential operator, integral equation

In recent years, the study of boundary value problems for loaded hyperbolic and mixed-type equations, along with their applications, has advanced significantly. Such equations play an important role in modeling physical and engineering processes, as they describe wave propagation, oscillations, and other dynamic phenomena. In particular, loaded hyperbolic equations with variable coefficients provide a more accurate representation of real systems, since spatial and temporal variations of parameters substantially affect the behavior of solutions.

In this paper, we investigate the existence and uniqueness of solutions to a boundary value problem for a fractionally loaded hyperbolic equation with variable coefficients. In addition, analytical solution methods based on integral equation techniques are developed.

The obtained results contribute to both the theoretical analysis and the applied modeling of complex hyperbolic systems [1,2].

In the first quadrant of the plane

$$G_\infty = (0, +\infty) \times (0, +\infty),$$

we consider the following initial-boundary value problem:

$$Lu(x,t) = u_{tt} - a^2(x,t)u_{xx} + b(x,t)u_t + c(x,t)u_x + d(x,t)u(x,t) =$$
$$= F(x,t) + \sum_{i=1}^{n} \lambda_i(x,t) D_{0\xi}^{\alpha_i} u(\xi, 0), \quad (x,t) \in G_\infty, \alpha < 0 \tag{1}$$
$$u(x,t)|_{t=0} = \psi_1(x),\, u_t(x,t)|_{t=0} = \psi_2(x), x > 0 \tag{2}$$
$$u(x,t)|_{x=0} = \gamma(t), t > 0 \tag{3}$$

where the coefficients a, b, c, d are real-valued functions, $\xi = x - t$, and $F, \psi_1, \psi_2, \gamma$ are given real-valued functions of their respectivevariables. $D_{0\xi}^{\alpha_i}$ denotes the fractional integration operator of order α_i with respect to the variable ξ, in the sense of Rieman-Lioville For problem (1)–(3), under suitable conditions on the given functions, the existence and uniqueness of a solution are established. The proof is based on the construction of the Riemann function for the corresponding unloaded hyperbolic equation. By exploiting properties of the Riemann function together with those of the fractional integration operator, the unique solvability of the loaded equation is established.

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On a Problem of Traffic Flow Speed Control in the Payne-Whitham Model*

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The report is dedicated to the problem of traffic flow control on a given section of road. Flow dynamics are described using a modified Payne-Whitham macroscopic model. A term has been added to the model, describing external sources of influx or outflow of vehicles. The exact value of this term is unknown; only the limits of its variation are specified. The goal of control is to ensure that the average speed of the traffic flow at a given point in time remains within a specified segment.

Keywords: control, traffic flow, macroscopic model

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Consider a road section of length l and a time segment $[0,T]$. Let $\rho(t,x)$ denote the traffic flow density (the number of cars per unit length) at time $t\in[0,T]$ and point $x\in[0,l]$, and let $v(t,x)$ denote the flow velocity. The dynamics of the traffic flow are described by the modified Payne-Whitham model [1]:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=f(t,x),\quad \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{1}{\tau}(V(\rho)-v), \tag{1}$$

where $V(\rho)$ is the speed considered preferable by drivers. Here $\tau>0$ is the relaxation time, characterizing the drivers' inertia. Unlike the classical Payne-Whitham model, we assume that this system is influenced by the function $f(t,x)$, which describes external sources of influx or outflow of vehicles. The function $f(t,x)$ is not precisely defined, but its estimates are known:

$$f_1(t,x)\le f(t,x)\le f_2(t,x),\quad (t,x)\in[0,T]\times[0,l], \tag{2}$$

where $f_1(t,x)$, $f_2(t,x)$ are continuous functions.

The initial conditions are given in the form

$$\rho(0,x)=\rho_0(x),\quad v(0,x)=v_0(x),\quad x\in[0,l], \tag{3}$$

where $\rho_0(x)$ and $v_0(x)$ are given continuous functions on the segment $[0,l]$.

At the left end of the road section ($x=0$), the flow density changes according to the equation

$$\frac{d\rho(t,0)}{dt}=c_1(t)+c_2(t)u(t),\quad |u(t)|\le 1,\quad t\in[0,T], \tag{4}$$

where $u(t)$ is the control; the functions $c_1(t),c_2(t)$ are continuous, and $c_2(t)\ge 0$ and

$$\int_0^t c_1(r)dr\ge\int_0^t c_2(r)dr,\quad 0\le t\le T,$$

which guarantees the non-negativity of the density $\rho(t,0)$.

The flow velocity at the left end is specified as follows: $v(t,0)=v_*(t),\quad t\in[0,T]$, where $v_*(t)$ is a given continuous function.

At the right end ($x=l$), we assume that the road has maximum capacity, which corresponds to the conditions:

$$\rho(t,l)=0,\quad \frac{\partial\rho(t,l)}{\partial x}=0,\quad \frac{\partial v(t,l)}{\partial x}=0,\quad v(t,l)=v_{\max},\quad 0\le t\le T. \tag{5}$$

The condition $\rho(t,l)=0$ corresponds to free exit, and the condition $v(t,l)=v_{\max}=\text{const}$ means that the exit speed is maximum.

Let $\varepsilon>0$ and $q>0$ be given. The goal of choosing a control $u(t)$ in (4) is to satisfy the inequality

$$\left|\int_0^l v(T,x)\sigma(x)dx-q\right|\le\varepsilon \tag{6}$$

for any continuous function (2). Here $\sigma:[0,1]\to\mathbb{R}$ is a given continuous function that satisfies the conditions $\sigma(0)=0$, $\sigma(x)\ge 0$ for $x\in(0,l]$.

After a change of variables, the problem (1)–(6) is reduced [2] to a one-dimensional single-type control problem in the presence of disturbances:

$$\dot z=-a(t)u+b(t)\eta,\quad |\eta|\le 1,\quad |z(T)|\le\varepsilon.$$

A set of initial positions is found for which there exists a control u that guarantees achievement of the goal (6) for any unknown functions (2).

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Regular Solutions in Sobolev Spaces for Degenerate Integro-Differential Equations

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This report presents new results on the solvability in classes of regular solutions of boundary value problems for integro-differential equations

Keywords: integro-differential equations, degeneracy, regular solutions, boundary value problems, Keldysh type degeneracy

This report presents new results on the solvability in classes of regular solutions of boundary value problems for the following types of integro-differential equations:

1. second-order integro-differential equation

$$h_0(t)u_{tt} + h_1(t)u_t + h_2(t)u - \int_0^t R(t,\tau)u(\tau)\,d\tau = g(t), \tag{1}$$

2. higher-order even-order integro-differential equation

$$h(t)D_t^{2p}u(t) + c(t)u(t) + \int_0^t R(t-\tau)u(\tau)\,d\tau = g(t), \tag{2}$$

3. higher-order odd-order integro-differential equation

$$h(t)D_t^{2p+1}u(t) + c(t)u(t) + \int_0^t R(t,\tau)u(\tau)\,d\tau = g(t), \tag{3}$$

All these equations exhibit degeneracy of the Keldysh type. In equation (1), the function $h_0(t)$ may vanish on the interval $[0,T]$, while in equations (2) and (3), the function $h(t)$ may vanish. Equation (1) in the case $h_0(t) \equiv 0$ — the boundary value problem for such an equation — was studied in [1]. For equations (2) and (3) in the multidimensional case in the absence of the integral term, the study was carried out in [2, 3].

In all cases, existence and uniqueness theorems are proved for solutions possessing all generalized in the sense of S. L. Sobolev derivatives appearing in the corresponding equation.

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Reduction of the Problem of Constructing the Lyapunov Matrix to a Fredholm Equation*

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In this work we consider a linear time-delay system with multiple delays, commensurate or not. It is shown that a Fredholm integral equation of the second kind with a piecewise constant kernel is solved by the Lyapunov matrix of this system. Thus, this matrix can be found numerically with a given accuracy. The result generalizes some results of the work Egorov, Aleksandrova (2025).

Keywords: Lyapunov matrix, Fredholm equation, delay systems

Consider a delay differential system

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t - h_j). \tag{1}$$

Here, the delays satisfy $0 = h_0 < ... < h_m = H$, and coefficients $A_j \in \mathbb{R}^{n\times n}$, $j \in \{0, \dots, m\}$.

Introduce some notations:

$$A = \sum_{j=0}^{m} A_j,$$

$$M = A^T \otimes I_n + I_n \otimes A^T.$$

Here $\otimes$ is Kronecker product, $I_n \in \mathbb{R}^{n\times n}$ is identity matrix.

Definition 1. A continuous function $U(t)$, $t \in [-H, H]$, is called the delay Lyapunov matrix of equation (1) associated with a given matrix $W > 0$, if it satisfies the following three properties:

$$U'(t) = \sum_{j=0}^{m} U(t - h_j)A_j, \quad t \in (0, H),$$

$$U(-t) = U^T(t), \quad t \in [-H, H],$$

$$\sum_{j=0}^{m} [U^T(h_j)A_j + A_j^T U(h_j)] = -W.$$

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The Lyapunov matrix is an important object in the Lyapunov-Krasovskii method [2], that is used in delay systems analysis.

If system (1) has only multiple commensurate delays, the well-known method is used [2]. The problem of this method is that it is very unreliable with a large number of delays. If the delays are incommensurate, the numerical methods are used. Several approaches exist that allow to compute an approximation of the Lyapunov matrix for a given problem. A method for exponentially stable systems is proposed in article [3]. A problem for a scalar equation is considered in paper [1]. The problem is reduced to a Fredholm integral equation and the numerical sheme is described. In this work some of the results presented in the article [1] are generalized for the matrix case. For a delay system, a corresponding system of integral Fredholm equations with piecewise constant kernel is constructed. Presented method does not require exponential stability of a system.

For further computation, we represent the Lyapunov matrix as a vector. We use the vectorization operator vec, which satisfies the following properties:

$$\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B),$$
$$\mathrm{vec}(A^T) = \mathcal{K}\mathrm{vec}(A),$$

where $\mathcal{K}$ is the commutation matrix [4]. Denote $u(t) = \mathrm{vec}(U(t))$, $w = \mathrm{vec}(\mathrm{W})$.

We use properties of Lyapunov matrix to prove the following fact:

Theorem. *If* $\det M \neq 0$*, then the Lyapunov matrix of system (1) is a solution to the equation*

$$u(\tau) = g(\tau) + \int_0^H f(\tau, s)u(s)ds,$$

where

$$f(\tau, s) = \sum_{j=0}^{m} \Big(\chi(\tau - h_j - s)(A_j^T \otimes I_n) - \chi(h_j - \tau - s)(A_j^T \otimes I_n)\mathcal{K}\Big) + \sum_{j=0}^{m} L_j \chi(h_j - s),$$

$$L_j = \Big[(A_j^T \otimes I_n)\mathcal{K} - M^{-1}(I_n + \mathcal{K})(A^T \otimes A_j^T)\Big],$$

$$g(\tau) = -M^{-1}w.$$

This equation can be solved numerically with a given estimate using Anselone's approach [5].

The research is carried on with support of RNF, project No. 23-71-10099.

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Nonlinear stability for the Vlasov-Poisson equations*

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We consider the first mixed problem for the Vlasov-Poisson system in the half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_1 > 0\}$. The energy-Casimir method is applied to proove the nonlinear stability of a stationary solution with zero potential and distribution functions compactly supported with respect to the hyperplane $x_1 = 0$.

Keywords: Vlasov equations, nonlinear stability, energy-Casimir method

1 The main results

For $t \in (0, T)$ in the half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_1 > 0\}$ and with $v \in \mathbb{R}^3$ we consider the Vlasov-Poisson system:

$$\frac{\partial f^\beta}{\partial t} + (v, \nabla_x f^\beta) + \frac{\beta e}{m_\beta}\left(-\nabla_x \varphi + \frac{1}{c}[v, B(x)], \nabla_v f^\beta\right) = 0, \tag{1}$$

$$-\Delta\varphi = 4\pi e \rho, \tag{2}$$

$$\rho = \int\limits_{\mathbb{R}^3} \sum_{\beta=\pm 1} \beta f^\beta(x, v, t) dv, \tag{3}$$

with initial conditions

$$f^\beta(x, v, 0) = f_0^\beta(x, v), \tag{4}$$

and the Dirichlet boundary condition

$$\varphi|_{x_1=0} = 0. \tag{5}$$

Here $f^\beta(x, v, t)$ are the distribution functions of charged particles, for $\beta = +1$ of positively charged ions, for $\beta = -1$ electrons, at point x with velocity v at time t. The function $\varphi(x)$ is the potential of self-consistent electric field. $B(x)$ is the induction of external magnetic field. Constants c, e, m_β represent respectively speed of light, electron charge, and mass of a charged particle (ion for $\beta = +1$ and electron for $\beta = -1$).

We apply the energy-Casimir method to study the nonlinear stability of a stationary solution to problem (1)-(5) with zero potential. There exist various concepts of stability for the Vlasov-Poisson system [2], [1]. Our analysis of nonlinear stability relies on the energy-Casimir method. In essence, the method consists of 4 fundamental steps, and the talk follows the structure below. The first step of the method is based on the fact that the total energy:

$$H(f^\beta) := \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3_+} \sum_{\beta=\pm 1} \frac{m_\beta}{2}|v|^2 f^\beta dx dv + \int\limits_{\mathbb{R}^3_+} |\nabla_x \varphi|^2 dx$$

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is a constant on any classical solution of the Vlasov-Poisson system. We use the results on the solvability of problem (1)-(5) obtained in [5]. The total energy is conserved for any classical solution of the characteristics equations for the Vlasov equation, regardless of the structure of the electric field potential. However, the stationary solution that we study is constructed for zero potential. In such a way the first integrals, which are used as arguments for a stationary solution with zero potential, will naturally no longer be the first integrals for a system of characteristics with a non–zero potential. Usually, this situation is solved assuming that the potential has some symmetry consistent with the geometry of the spatial domain. In the case of the problem in a cylindrical domain or the problem in R^3, radial or spherical potential symmetry is usually used. In our case, for the possibility of using the energy-Casimir method in the half-space $x_1 > 0$, we use the following conditions for the potential structure: we will assume that the potential does not depend on the variable x_2. In this case, we can get the first integral that depends on the variable x_1 and use it as an argument for a stationary solution. Thus, we obtain the solution having a compact support with respect to the variable x_1 in the half-space $x_1 > 0$.

On the second step we construct a functional C, which would also be constant in time for any classical solution of the Vlasov equations, so that stationary solution would be the critical point of the functional $H_C := H + C$. This means that the first variation of H_C should be equal to zero on our stationary solution. Further, on the third and fourth steps we get the lower and upper estimates for H_C. The lower estimates give us the positive definiteness of the second variation of H_C. The upper estimates give us a norm that will be used to obtain the nonlinear stability of our stationary solution.

This work is largely based on [3] and [4]. However unlike [3] and [4], we consider the problem in a half-space and we use sufficient conditions for the existence of a solution to problem (1)-(5) from [5]. We do not use the elastic reflection conditions since the magnetic field is sufficient to keep the characteristics at a certain distance from the boundary of the half-space [5].

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Boundary Value Problems in Hopfield-type Models of Neural Systems in the Case of Discontinuous Activation Functions of Neurons and Signal Delays*

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A system of delay differential equations is considered, which is a Hopfield-type model of a neural network with a discontinuous activation function of neurons. A two-point boundary value problem is studied. Statements on its solvability and on the monotone dependence of the solution set with respect to the threshold value of neuron activation, external inputs, equation coefficients, and boundary condition values are formulated.

Keywords: neural network, differential equation with discontinuous right-hand side, delay, boundary value problem, dependence of solutions on parameters

1 Notation and problem statement

We assume that the «usual» order is defined in $\mathbb{R}^n$: for vectors $x = (x_i)_{i=1,n}$, $y = (y_i)_{i=1,n}$ the inequality $x \le y$ is equivalent to $x_i \le y_i$ for $i = 1, n$. By L^n we denote the space of (Lebesgue) summable functions $[0,T] \to \mathbb{R}^n$ with the corresponding order: for $x, y \in L^n$ the relation $x \le y$ holds if and only if $x(t) \le y(t)$ for a.e. $t \in [0,T]$. By AC^n we denote the space of absolutely continuous functions $[0,T] \to \mathbb{R}^n$. Let a measurable function $h : [0,T] \to \mathbb{R}$ be given. For $u \in AC^1$ we define

$$(S_h u)(t) = \begin{cases} u(h(t)), & h(t) \in [0,T], \\ 0, & h(t) \notin [0,T]. \end{cases}$$

The function $S_h u$ defined in this way is obviously measurable and essentially bounded for any continuous function u, in particular for $u \in AC^1$.

In the paper, a Hopfield-type model of a neural network is considered, which is a system of delay differential equations of the form

$$(\mathcal{L}_i v_i)(t) := \dot{v}_i(t) + \alpha v_i(t) = \sum_{j=1}^{n} w_{ij} \chi_\theta((S_{h_{ij}} v_j)(t)) + J_i(t), \quad i = \overline{1,n}, \quad t \in [0,T], \tag{1}$$

with respect to the function $v = (v_i)_{i=\overline{1,n}} \in AC^n$ of the values at each time t of the electrical potentials of all n neurons of the network. Here $\chi_\theta(x) = \begin{cases} 0, & x < \theta, \\ 1, & x \ge \theta, \end{cases}$ — this is the activation function of neurons, its parameter $\theta > 0$ is the threshold value of the electrical potential (upon reaching which the neuron is activated); for any $i, j \in \overline{1,n}$, $i \ne j$ the functions $h_{ij} : [0,T] \to \mathbb{R}$

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are measurable, $h_{ij}(t) \le t$ for a.e. $t \in [0,T]$ ($t - h_{ij}(t)$ is the transmission time of the electrical impulse from the j-th neuron to the i-th neuron); $J_i \in L^1$ are external inputs, $J_i(t) \ge 0$ for a.e. $t \in [0,T]$; numerical coefficients $\alpha > 0$, $w_{ij} > 0$ and $w_{ii} = 0$.

For equation (1) we consider a boundary value problem with the condition

$$lv := v(0) - \beta v(T) = \gamma, \tag{2}$$

where $\gamma \in \mathbb{R}^n$. Boundary value problems with conditions (2) make it possible to model certain regimes and rhythms of electrical activity of the brain. For models without delay such problems were studied in [1].

2 The main results

The right-hand side of equation (1) is not continuous with respect to the phase variable, which complicates the application of methods of analysis, operator theory in normed and metric spaces, as well as classical results of the theory of differential equations. In the present study, results on the existence of fixed points in partially ordered spaces (the well-known Knaster–Tarski and Birkhoff–Tarski theorems) and their dependence on a parameter [1] are applied.

The set of solutions (in the space AC^n) of the boundary value problem (1), (2) will be denoted by $\mathcal{R}$. It is easy to see that if $\mathcal{R} \ne \emptyset$, then for any $v = (v_i)_{i=\overline{1,n}} \in \mathcal{R}$ the inequalities $\varsigma_i \le \mathcal{L}_i v_i \le \xi_i$, $i = \overline{1,n}$, hold, where $\varsigma_i(t) := J_i(t)$, $\xi_i(t) := J_i(t) + \sum_{j=1}^n w_{ij}$, $t \in [0,T]$. Using these estimates, on the basis of fixed point theorems for monotone operators, we obtain the following statement.

Theorem 1. *Let the inequalities* $0 \le \beta_i < \exp(\alpha T)$, $i = \overline{1,n}$, *hold. Then* $\mathcal{R} \ne \emptyset$, *moreover, the set* $\mathcal{L}\mathcal{R}$ *contains the greatest and the least elements.*

Now we study the dependence of the set $\mathcal{L}\mathcal{R}$ on the parameters of problem (1), (2). Let the following be given: an increasing sequence $\theta_k \subset \mathbb{R}$ and decreasing sequences $(w_{ij}^k)_{i,j=\overline{1,n}} \subset \mathbb{R}^{n\times n}$, $\gamma^k \subset \mathbb{R}^n$, $(J_i^k)_{i=\overline{1,n}} \subset L^n$, for which the relations $\sup \theta_k = \theta$, $\inf w_{ij}^k = w_{ij}$ for all $i,j = \overline{1,n}$, $i \ne j$, and $w_{ii}^k = 0$, $\inf \gamma^k = \gamma$, $\inf J_i^k(t) = J_i(t)$ for a.e. $t \in [0,T]$ hold.

Along with problem (1), (2) we consider a sequence of problems

$$(\mathcal{L}_i v_i)(t) = \sum_{j=1}^n w_{ij}^k \chi_{\theta_k}((S_{h_{ij}} v_j)(t)) + J_i^k(t), \quad i = \overline{1,n}, \quad t \in [0,T], \quad lv = \gamma^k, \tag{3}$$

$k = 1, 2, \ldots,$. The set of solutions (in AC^n) of the boundary value problem (3) will be denoted by $\mathcal{R}^k$. According to Theorem 1, $\mathcal{R}^k \ne \emptyset$. Moreover, it is easy to see that for each k and any $v^k = (v_i^k)_{i=\overline{1,n}} \in \mathcal{R}^k$ the inequalities $\varsigma_i \le \mathcal{L}_i v_i^k \le \xi_i^1$, $i = \overline{1,n}$, hold, where $\varsigma_i(t) := J_i(t)$, $\varsigma_i(t) := J_i(t)$, $\xi_i^1(t) := J_i^1(t) + \sum_{j=1}^n w_{ij}^1$, $t \in [0,T]$. Using these estimates and on the basis of the results of [1], we obtain the following statement.

Theorem 2. *For any* k_0 *and any* $v^{k_0} \in \mathcal{R}^{k_0}$, *for any* $k \ne k_0$ *one can choose* $v^k \in \mathcal{R}^k$ *such that the sequence* $\{\mathcal{L}v^k\}_{k\in\mathbb{N}}$ *is decreasing and* $\inf_{k\in\mathbb{N}} \mathcal{L}v^k \in \mathcal{L}\mathcal{R}$.

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Conditions and Methods of the Maximum Principle Based on Fixed Point Problems[*]

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In the class of optimal control problems linear in control, new forms of the known maximum principle conditions are developed as special fixed point problems in the control space. Based on the obtained fixed point problems, new iterative methods for finding extremal controls are constructed. The proposed fixed point methods are characterized by the property of improving the objective functional without the procedure of varying control approximations at each iteration, unlike known gradient-type methods. Convergence conditions for the constructed iterative processes are given. An illustrative example of finding extremal controls is provided.

Keywords: optimal control problem, maximum principle, fixed point problem, iterative method

The following optimal control problem is considered:

$$(u) = \varphi(x(t_1)) + \int_T F(x(t), u(t), t)dt \to \inf_{u \in V}, \tag{1}$$

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x^0, \quad u(t) \in U, \quad t \in T = [t_0, t_1], \tag{2}$$

in which $x(t) = (x_1(t), \dots, x_n(t))$ is the state of the system, and $u(t) = (u_1(t), \dots, u_m(t))$ is the control. The set of admissible controls is considered to be the set of piecewise-continuous functions taking values in a convex compact set $U \subset \mathbb{R}^m$: $V = \{v \in PC(T) : v(t) \in U,\ t \in T\}$.

The initial state x^0 and the time interval T are given. The function $\varphi(x)$ is continuously differentiable on $\mathbb{R}^n$. The functions $F(x, u, t)$ and $f(x, u, t)$, as well as their derivatives $F_x(x, u, t)$ and $f_x(x, u, t)$, are linear with respect to the variable u and continuous with respect to the set of arguments on the set $\mathbb{R}^n \times U \times T$. The function $f(x, u, t)$ satisfies the Lipschitz condition with respect to x in $\mathbb{R}^n \times U \times T$ with a constant $L > 0$:

$$\|f(x, u, t) - f(y, u, t)\| \le L\|x - y\|.$$

Let us consider the Pontryagin function with the adjoint variable $p \in \mathbb{R}^n$:

$$H(p, x, w, t) = \langle f(x, w, t), p\rangle - F(x, w, t).$$

We define the modified adjoint system in the following form:

$$\dot{p}(t) = -H_x(p(t), x(t), w(t), t) - r(t), \quad p(t_1) = -\varphi_x(x(t_1)) - q, \tag{3}$$

$$H(p(t), y(t), w(t), t) - H(p(t), x(t), w(t), t) = \langle H_x(p(t), x(t), w(t), t) + r(t), y(t) - x(t)\rangle, \tag{4}$$

$$\varphi(y(t_1)) - \varphi(x(t_1)) = \langle \varphi_x(x(t_1)) + q, y(t_1) - x(t_1)\rangle, \tag{5}$$

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in which, by definition, we set $r(t) = 0$ in the case where the functions F and f are linear with respect to x, and also in the case $y(t) = x(t)$. Similarly, $q = 0$ in the case of the linearity of the function φ with respect to x, and also in the case $y(t_1) = x(t_1)$.

In problem (1), (2), linear in state, the modified adjoint system (3)–(5) coincides by definition with the standard adjoint system with the adjoint variable $\psi \in \mathbb{R}^n$:

$$\dot{\psi}(t) = -H_x(\psi(t), x(t), w(t), t), \quad \psi(t_1) = -\varphi_x(x(t_1)).$$

In problem (1), (2), nonlinear in state, the algebraic equations (4) and (5) can always be resolved with respect to the variables $r(t)$ and q (possibly non-uniquely).

For the controls $u \in V$ and $v \in V$, let us denote by $p(t, u, v)$, $t \in T$, the solution of the modified adjoint system (3)–(5) given $x(t) = x(t, u)$, $y(t) = x(t, v)$, and $w(t) = u(t)$. From the definition, the obvious equality $p(t, u, u) = \psi(t, u)$, $t \in T$ follows, where $\psi(t, u)$ is the solution of the standard adjoint system.

Using the mapping based on the projection operation with the parameter $\alpha > 0$:

$$u^\alpha(\psi, x, u, t) = P_U(u + \alpha H_u(\psi, x, u, t)), \quad \psi \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \quad t \in T,$$

the known necessary optimality condition for the control $u \in V$ (the maximum principle condition) [1] can be represented in the following form:

$$u(t) = u^\alpha(\psi(t, u), x(t, u), u(t), t), \quad t \in T. \tag{6}$$

In this paper, a new form of condition (6) is considered as the following system:

$$u(t) = u^\alpha(p(t, u, v), x(t, v), u(t), t), \quad v(t) = u^\alpha(p(t, u, v), x(t, v), u(t), t), \quad t \in T. \tag{7}$$

Another equivalent form of condition (6) is represented as the system:

$$u(t) = u^\alpha(p(t, v, u), x(t, u), u(t), t), \quad v(t) = u^\alpha(p(t, v, u), x(t, u), u(t), t), \quad t \in T. \tag{8}$$

Conditions (7) and (8) are considered as special fixed point problems, for the solution of which the corresponding iterative processes with the index $k \geq 0$ are constructed:

$$u^{k+1}(t) = u^\alpha(p(t, u^k, v), x(t, v), u^k(t), t), \quad v(t) = u^\alpha(p(t, u^k, v), x(t, v), u^k(t), t), \quad t \in T.$$
$$u^{k+1}(t) = u^\alpha(p(t, v, u^k), x(t, u^k), u^k(t), t), \quad v(t) = u^\alpha(p(t, v, u^k), x(t, u^k), u^k(t), t), \quad t \in T.$$

The iterative processes are characterized by the improvement property:

$$(u^{k+1}) - (u^k) \leq -\frac{1}{\alpha} \int_T \|u^{k+1}(t) - u^k(t)\|^2 dt.$$

Under certain assumptions, the convergence of the specified iterative processes in terms of the value $\delta(u^k) = (u^k) - (u^{k+1})$, characterizing the residual of the maximum principle on the control u^k, is proved.

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An Approach to Constructing Matryoshka Multistable Dynamical Systems

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The paper proposes a method for constructing an extremely multistable dynamical system containing a countable number of nested self-similar chaotic attractors.

Keywords: extreme multistability, matryoshka multistable system, Lyapunov exponents, Kaplan–Yorke dimension

1 The main results

Dynamical systems possessing an infinite number of coexisting attractors are called extremely multistable (megastable if the number of attractors is countable). Some methods for artificial generation of extremely multistable chaotic systems have been proposed in [1,2]. A new type of extreme multistability was introduced recently —matryoshka multistability. Matryoshka multistability is a special type of megastability characterized by an infinite number of nested self-similar chaotic attractors converging to zero. In other words, the set of attractors of the system forms a structure in its phase space similar to a Russian nesting doll. The papers [3,4] showed how, for example, to create matryoshka-multistable systems with an odd nonlinear function using well-known systems in Lurie form.

In this paper, a matryoshka-multistable system is constructed on the basis of the system with even nonlinearity

$$\begin{cases} \dot{x} = -z, \\ \dot{y} = -y + \varphi_0(x), \\ \dot{z} = 1.7x + y, \\ \varphi_0(x) = 1.7 - x^2. \end{cases} \tag{1}$$

Let $X = \mathrm{col}(x, y, z)$ and $f(X) = \mathrm{col}(-z,\ -y - x^2 + 1.7,\ 1.7x + y)$. System (1) has two equilibrium states $X_1 = (x_1,\ -1.7x_1,\ 0)$ and $X_2 = (x_2,\ -1.7x_2,\ 0)$, where $x_{1,2}$ are the roots of the equation $x^2 - 1.7x - 1.7 = 0$. Both equilibrium states are unstable. For system (1), $\mathrm{div}(f(X)) = -1$; therefore, the system under consideration is dissipative. From the neighborhood of both equilibrium states of the system, the same chaotic attractor is excited, with its Lyapunov exponents are $(0.044,\ 0,\ -1.044)$ and with its Kaplan–Yorke dimension is 2.042. System (1) has one remarkable property.

Let $k \in \mathbb{Z}$ be an arbitrary integer and $m > 0$ an arbitrary real number. If, in system (1), the nonlinearity $\varphi_0(x)$ is replaced by the function

$$\varphi_k(x) = m^k \varphi_0\left(\frac{x}{m^k}\right), \tag{2}$$

then the new system also has a chaotic attractor, which can be visualized by numerical integration, for example, with the initial condition $m^k X_1$.

Basing on the given property of system (1) and on appropriately choosing the number m, it is possible to construct a matryoshka multistable system on its basis. Let $m = 15$ and set $\alpha_k = 3m^k$. We define the function $g(x)$ as follows: for an arbitrary integer $k_0 \in \mathbb{Z}$ on the interval $(\alpha_{k_0}, \alpha_{k_0+1}]$ we set $g(x) = \varphi_{k_0+1}(x)$, where $\varphi_k(x)$ is given by formula (2).

Proposition. Let

$$\varphi(x) = \begin{cases} g(x), & x \geq 0, \\ g(-x), & x \leq 0. \end{cases}$$

If the nonlinearity $\varphi_0(x)$ in system (1) is replaced by the function $\varphi(x)$, the resulting system will be a matryoshka multistable system, i.e., it will contain a countable number of self-similar chaotic attractors converging to zero. All attractors will have identical Lyapunov exponents and the same Kaplan–Yorke dimension.

Note that the nonlinearity $\varphi(x)$ has a countable number of discontinuity points of the first kind, and system (1) with such a nonlinearity has a countable number of equilibrium states. In this case, the solution of a system with discontinuous nonlinearity should be considered according to Filippov [5].

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On the Design of a Control Scheme with a Guide under Conditions of Uniqueness and Uniform Boundedness of Trajectories Generated by Control Measures

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A nonlinear approach-evasion differential game (DG) on a finite time interval is considered for a control system satisfying the conditions of uniqueness and uniform boundedness of generalized trajectories. A variant of a control scheme with a guide (model) is presented, defined based on a multivalued non-anticipatory strategy implemented in control-measure spaces. This scheme is used to prove an alternative theorem in the case of phase constraints with closed time sections, but not closed, generally speaking, in the coordinate-wise convergence topology. A theorem on the alternative solvability of DG in classes of control procedures that do not use the values of the adversary's controls is provided.

Keywords: alternative solvability, guided control, non-anticipatory strategies

The question of implementing an alternative partition of the DG space is related to the construction of a positional absorption set – the maximal stable bridge of N.N. Krasovsky and A.I. Subbotin [1,2] (in this paper, we adhere to the extended formulation proposed by A.V. Kryazhimsky [3,4]). An important role in this construction was played by backward procedures [5] based on ideas from dynamic programming, as well as program constructions under regularity conditions (see [2,6]). In the general case of nonlinear DG, the method of program iterations (MPI) was used (see [8,9,10] and others).

In this study, we focus on the use of non-anticipatory control strategies, such as those constructed within the framework of the MPI [13] or the scheme [14], in the design of a guide. This combination allows us to extract the solution to a positional control problem from the solution of logically simpler program control problems, as well as avoid unrealistic assumptions about the admissibility of the presence of the opponent's current control values in the input data of the ally's non-anticipatory strategy.

The second feature of the proposed approach is the rejection of the local Lipschitz property of the original control system with respect to the phase variable in favor of A.V. Kryazhimskii's conditions (uniform boundedness and uniqueness of motions under relaxed control). Moreover, the z-motions proposed by A.V. Kryazhimskii arise in the guide's design as a link between the motions of the original system and the optimal model [4, c. 30], [7] (see also [11,12]).

The paper provides control scheme with the guide that includes three motions: the motion of system under consideration – real system – called x-motion, the above z-motion and a model motion with dynamic identical to the dynamic of real system (w-motion):

– x-motion is governed by a real disturbance v and by the realization u^e of ordinary control to track each other with z-motion in the procedure of extremal aiming;

– w-motion is generated by the relaxed control η provided by an optimal non-anticipating multivalued strategy as an “answer” to the embedding into corresponding set of measures of ordinary control v^e arising in the extremal aiming procedure between x-motion and z-motion;

– z-motion is governed by the same relaxed control η and by the x-motion.

The indicated scheme guarantees the solution of the approach problem in the initial positions for which this problem is solved in the class of non-anticipating multivalued strategies. At the same time, with the specified design of the ally player's strategy, data on the current control values of the opposing player is not used.

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On Numerical Experiments for Solving Initial Value Problems for Singular Systems of Quasi-Linear Volterra Integral Differential Equations of General Form

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We study singular systems of quasi-linear Volterra integral differential equations. The concepts of index, solution manifold dimension, and singular points are formalized. Numerical methods for solving initial value problems are described, and experimental results are discussed.

Keywords: integral differential equations, Volterra equations, index, solution manifold, singular points

We consider systems of integral differential equations of the form

$$\Lambda_1(z, Vz) := A(z, Vz, t)\dot{z} + B(z, Vz, t) = 0, \ t \in T = [\alpha, \beta], \tag{1}$$

where $A(\xi, \zeta, t)$ is a given $(n \times n)$-matrix, $B(\xi, \zeta, t)$ is a given vector function, $\xi \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m$, $z \equiv z(t)$ is the unknown vector function, $\dot{z}(t) = dz(t)/dt$, $Vz = \int\limits_{\alpha}^{t} K(t, s, z(s))ds$, $K(t, s, \xi)$ is an m-dimensional vector function. It is assumed that the matrix $A(\xi, \zeta, t)$ and the vector functions $B(\xi, \zeta, t)$, $K(t, s, \xi)$ are sufficiently smooth in their domains of definition and system (1) satisfies the condition

$$\det A(\xi, \zeta, t) = 0 \ \forall (\xi, \zeta, t) \in \mathbb{R}^n \times \mathbb{R}^m \times T, \tag{2}$$

By a solution to (1) we mean any vector function $z(t)$ differentiable on $T_\alpha = [\alpha, \alpha + \delta] \subseteq T$ that turns (1) into an identity on T_α.

When studying systems of the form (1), the Cauchy problem is usually posed:

$$z(\alpha) = b, \tag{3}$$

where b is a given vector from $\mathbb{R}^n$. The research underlying this report makes essential use of the results that can be found in [1].

For a solution of the Cauchy problem for system (1) to exist, it is necessary — though not in general sufficient — that the Kronecker–Capelli criterion hold at $t = \alpha$ for vector $z^{(1)}(\alpha)$:

$$\text{rank } A(b, 0, \alpha) = \text{rank} A(b, 0, \alpha)| - B(b, 0, \alpha) \tag{4}$$

Therefore, initial value problems for (1) can be posed below as

$$\mathrm{P}z(\alpha) = \mathrm{b}, \tag{5}$$

where a given $(\nu \times n)$-matrix P is full rank, vector b is set and

$$\text{rank } \mathrm{P} = \nu, \ \nu \leq n.$$

Problem (1), (5) with $\mathrm{P} = E_n$, where E_n is the identity matrix of dimension n, coincides with the Cauchy problem, $\mathrm{b} = b$.

In this talk, we address numerical methods for solving initial value problems (1), (5) and discusses the results of numerical experiments performed in the presence of singular points in the domain.

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Sparse Minimizing in Hardy Spaces of Holomorphic Functions*

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We consider the Hardy space $\mathcal{H}^p(D)$, $1 < p < +\infty$, of holomorphic functions on bounded simply connected plane domain D as a Banach space with reproducing kernel. For $p = 2$ the reproducing kernels of the Hilbert spaces $\mathcal{H}^2(D)$ were identified for a number of domains of special types (a disc, a half plane and so on). However, for general plane domain there are no simple formulae for the kernels. Using deep connection of the natural pairing between the Lebesgue spaces $L^p(\partial D)$ and $L^q(\partial D)$, $1/p + 1/q = 1$, with the classical Grothendieck-Köthe-Silva duality, we identify the reproducing kernel of the space $\mathcal{H}^p(D)$ as the Cauchy kernel. As an application of this fact we investigate probem of the so-called sparse minimizing of proper convex coercive lower semi-continuous functionals on the Hardy spaces $\mathcal{H}^p(D)$ related to finite-dimensional space of data.

Keywords: Banach space with reproducing kernel, Hardy space, duality theorems, sparse minimizing

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On the Solvability of the Cauchy Problem for Pseudohyperbolic Equations in Weighted Sobolev Spaces*

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A class of strictly pseudohyperbolic equations is considered. Conditions for the unique solvability of the Cauchy problem in the class of weighted Sobolev spaces are established and estimates for solutions are obtained.

Keywords: equations not solvable with respect to the highest derivative, pseudohyperbolic equations, weighted Sobolev spaces, energy estimates

In the monograph [1], we introduced a certain classification of linear partial differential equations of the following form

$$L_0(D_x)D_t^l u + \sum_{k=0}^{l-1} L_{l-k}(D_x)D_t^k u = f(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{1}$$

and a wide class of boundary value problems has been studied for them. Such equations are often called ***Sobolev type equations***, because the research of S.L. Sobolev [2] was the first in-depth studies of differential equations not solvable with respect to the highest derivative. Currently, there is a large number of theoretical and applied works devoted to the study of various problems for equations of the form (1). However, for the class of pseudohyperbolic equations introduced in [1], the theory of boundary value problems is still little studied. It should be noted that even in the case of constant coefficients for the Cauchy problem for pseudohyperbolic equations, there are a number of features. Namely, as shown in [1], [3], for the solvability of the Cauchy problem in Sobolev spaces $W_2^m((0,T)\times\mathbb{R}^n)$, the right part $f(t,x)$ of the equation must have additional smoothness and depend on the orders of differential operators and dimension n must be orthogonal to some monomials x^β.

In this paper, we continue to study the Cauchy problem for pseudohyperbolic equations, but in the broader class of Sobolev spaces with special weights. In these spaces, it is possible to identify a wider class of pseudohyperbolic equations compared to [1], for which conditions for the unique solvability are established. These conditions are analogues of the conditions for the solvability of the Cauchy problem for strictly hyperbolic equations (see [4], [5]).

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Vekua's Formula and the Construction of Eigenfunctions and Eigenvalues of the Laplace Operator

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An algorithm for constructing eigenfunctions and eigenvalues of the Laplace operator in n-dimensional star-shaped domains is presented using Vekua's formula relating solutions of the Laplace equation and the Helmholtz equation.

Keywords: Vekua's formula, eigenfunctions, Laplace operator, star-shaped domains

As is well known (see, for example, [1]), in a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\Gamma = \partial\Omega$, the Laplace operator Δ has a discrete spectrum consisting of eigenvalues λ_k with the corresponding eigenfunctions $V_k \in C(\Omega)$ such that

$$-\Delta V_k = \lambda_k V_k \qquad \text{and} \qquad 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \to \infty \quad \text{as} \quad k \to \infty\,,$$

Below we present the main steps of an algorithm for the numerical computation of eigenfunctions and eigenvalues of the Laplace operator. In contrast to the existing methods based on discretization of the Laplace operator, this algorithm relies on Vekua's formula [2]. Its elementary proof is given in [3], while the formula itself is stated in the following theorem.

Theorem 1. *Let $n \geq 2$, let the bounded domain $\Omega \subset \mathbb{R}^n$ be star-shaped with respect to the origin, and let the function $u : \Omega \ni x \mapsto u(x)$ be harmonic in Ω and continuous up to the piecewise smooth boundary $\Gamma = \partial\Omega$ of this domain. Then, for $\lambda > 0$, the function $v[u] = v_\lambda[u]$ defined by*

$$v[u](x) = u(x) - \int_0^1 \sum_{j\geq 0} (-1)^j \frac{t^{n/2}}{t}\Big(\lambda\frac{|x|^2}{4}\Big)^{j+1} \frac{(1-t)^j}{j!(j+1)!} u(tx)\, dt\,, \tag{1}$$

satisfies the Helmholtz equation $\Delta v[u] + \lambda v[u] = 0$.

It is reasonable to compute the numbers $\lambda_k > 0$ and the functions v_{λ_k} for $k \le K$, with some K, in descending order of $k \le K$. This is due to the following two facts noted in [4].
1). As $k \to \infty$, the asymptotic relation $\sigma_k - \sqrt{\lambda_k} = o(k^{-\infty})$ holds, where $\{\sigma_k\}_{k \ge 1}$ are the eigenvalues of the Dirichlet–Neumann operator, which for $n = 2$ may be regarded as known (cf. [5]).
2). If $\Omega \subset \mathbb{R}^2$, then there exists a constant $C = C(\Omega) > 0$, common to all small perturbations of the domain Ω, such that $|\sigma_k - \sqrt{\lambda_k}| < C, \quad k \ge 1$. This localization facilitates the search for λ_k taking into account the values λ_j with $j > k$ found previously.

Refinement of the values λ_k and the computation of the corresponding eigenfunctions are based on the following obvious proposition.

Proposition 1. *If $F \in C(\Gamma)$ and the function $u = u^F$ satisfies*

$$\Delta u = 0 \quad in \quad \Omega, \qquad u\Big|_{\Gamma} = F, \tag{2}$$

then the function $v_\lambda^F = v_\lambda[u^F]$ defined by formula (1) *is an eigenfunction of the operator $-\Delta$ (called the Laplace operator) that vanishes on Γ if and only if $v_\lambda^F\Big|_{\Gamma} = 0$.*

Using formulas for solving problem (2) (such numerically implementable formulas in the case $n = 2$ were obtained in [6] by the algorithm of [7]- [8]), the desired eigenfunctions $v_{\lambda_k} = v_{\lambda_k}^F$ are found numerically by minimizing a convex functional in a finite-dimensional space through the choice of a suitable finite-dimensional control parameter. This control parameter is the set of the first coefficients in the Fourier expansion of the function F, while the minimized functional is the sum of squares of the first coefficients in the Fourier expansion of the function $v_{\lambda_k}^F\Big|_{\Gamma}$.

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About One Class of Nonlinear High-dimensional Systems of Differential Equations

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The Cauchy problem for a system of nonlinear ordinary differential equations of large dimension with non-zero initial data is considered. We investigate asymptotic properties of solutions to the problem in dependence of the growth of the number of the equations. We prove that, for sufficiently large number of differential equations, the last component of the solution to the Cauchy problem is an approximate solution to an initial value problem for one delay differential equation.

Keywords: system of ordinary differential equations of large dimension, asymptotic properties of solutions, delay differential equation

The following Cauchy problem for the system of nonlinear ordinary differential equations is considered

$$\begin{cases} \dfrac{dz_1}{dt} = g(t, z_n) - \dfrac{n-1}{\tau_1} z_1 + \dfrac{n-1}{\tau_2} z_2, \qquad t > 0, \\ \dfrac{dz_j}{dt} = \dfrac{n-1}{\tau_1} z_{j-1} - \left(\dfrac{n-1}{\tau_1} + \dfrac{n-1}{\tau_2} \right) z_j + \dfrac{n-1}{\tau_2} z_{j+1}, \quad j = 2, \dots, n-2, \\ \dfrac{dz_{n-1}}{dt} = \dfrac{n-1}{\tau_1} z_{n-2} - \left(\dfrac{n-1}{\tau_1} + \dfrac{n-1}{\tau_2} \right) z_{n-1}, \\ \dfrac{dz_n}{dt} = -\theta z_n + \dfrac{n-1}{\tau_1} z_{n-1}, \\ z|_{t=0} = z_0, \end{cases} \tag{1}$$

where $\theta > 0$, $\tau_2 > \tau_1 > 0$. Here function $g(t, u) \in C(\overline{\mathbb{R}}_+^2)$ is bounded and satisfies Lipschitz condition; i.e.

$$|g(t, u_1) - g(t, u_2)| \le L|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R}, \quad t > 0.$$

We study properties of solutions to (1) with some initial vector z_0 for $n \gg 1$. We obtain estimates characterizing asymptotic behaviour of $z_n(t)$ as $n \to \infty$. Using methods proposed by G.V. Demidenko (see, for instance, [1, 2]), we prove closeness of $z_n(t)$ for $n \gg 1$ and a solution $y(t)$ to the following delay differential equation

$$\frac{dy}{dt} = -\theta y + g(t - \tau, y(t - \tau)), \tag{2}$$

where $\tau = \frac{\tau_1 \tau_2}{\tau_2 - \tau_1}$. This paper continues our investigation in [3] and will be published in [4].

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Analysis of the Critical State of the Plastic Strip with Variable Strength along the Less Durable Layer when Stretched

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The stress state in a stretchable plastic strip under plane deformation is studied. The strip is discretely heterogeneous along longitudinal direction – it contains a transverse rectangular layer of less durable material, and it is continuously heterogeneous across. In particular, the layer is continuously heterogeneous along longitudinal direction. It is assumed that the tangential stresses in a certain (small) neighborhood of the longitudinal axis of symmetry of the strip depend linearly on the distance to this axis. Explicit analytical expressions have been obtained for the approximate calculation of the critical values of tangential and normal stresses at the boundary between the layer and the rest of the strip material, explicit analytical expressions for calculating the critical load depending on the geometric and mechanical parameters of the strip.

Keywords: plastic layer, plane deformation, heterogeneous joint, stress state, critical load

1 The main results

The crticle considers (in dimensionless values) a strip containing an interlayer $\{(x;y) : x \leq 1\}$. It is assumed that the tangential stresses in the vicinity of the longitudinal axis Oy are linear in x, and the inhomogeneity function has the form $\{(x;y) : x \leq 1, |y| \leq d, d < 0,5\}$. Main results:
1) Approximate analytical expressions for critical stresses are obtained and the formula for the average limiting force is derived.

$$\tau_{xy} = (K-1)Z$$

$$\sigma_x = -\frac{a^2x^2}{ch^2(2ay)} + \sigma_F + a^2x_F^6 - 2Z(x_F)$$
$$\sigma_y = -a^2x^2 + 2Z(x) + \sigma_F + a^2x_F^2 - 2Z(x_F),$$

where τ, σ_x, σ_y - tangential and normal stresses, F is the point of exit of the characteristic from the corner of the layer to the contact surface; x_F is the abscissa of the point $F, \tau_F = \tau_{xy}(x_F, d), \sigma_F = \sigma_y(x_F, d)$, where d - layer thickness. Value $x_F = 1 - \dfrac{4d}{1+K}$ derived in [1], $\tau_F = (K-1)Z(x)F), \sigma_F = \dfrac{1+4K-K^2}{2}Z(x_F)$ derived in [2].

2) It is noted that as the thickness of the interlayer decreases, the limiting force increases. The average critical load (average critical voltage) is calculated using the formula:

$$\sigma_{av} = \int_a^b \sigma_y(x,d)dx,$$

where

$$\sigma_y = \min(-a^2x^2 + 2Z(x) + \sigma_F + a^2x_F^2 - 2Z(x_F, 2KZ(x)), x \in [0, x_F]$$
$$\sigma_y = \sigma_F Z(x), x \in [x_F, 1].$$

It follows that the critical load point increases with decreasing thickness of the interlayer, but does not reach the critical load for the strip of base material.
3) Even a slight heterogeneity of the layer material significantly affects the critical load.

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Quotient Equation for the One-dimensional Inviscid Flow along One Class of Curves

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We consider the Euler system describing a one-dimensional inviscid flows in space along curves of a certain class. Using differential invariants for the Euler system, the quotient equation is obtained. The solutions of the quotient equation that are constant along characteristic vector field provide some solutions of the Euler system.

Keywords: Euler system, differential invariants, quotient equation

1 The main results

We consider the Euler system [1] of PDEs on a space curve $\{x = f(a),\ y = g(a),\ z = \lambda a^2\}$, where λ is a positive constant:

$$\begin{cases} \rho(v_t + vv_a) = -p_a - 2\rho g\lambda a, \\ \rho_t + (\rho v)_a = 0, \\ \rho\theta(s_t + vs_a) - k\theta_{aa} = 0, \end{cases} \tag{1}$$

where $v(t,a)$ is the flow velocity, $p(t,a)$, $\rho(t,a)$, $s(t,a)$, $\theta(t,a)$ are the pressure, density, specific entropy, temperature of the fluid respectively, k is the constant thermal conductivity, g is the gravitational acceleration.

A Lie algebra $\mathfrak{g}$ of point symmetries of the PDE system (1) is generated by the vector fields

$$\begin{aligned} &X_1 = \partial_t, \quad X_2 = \partial_p, \quad X_3 = \partial_s, \quad X_4 = \theta\,\partial_\theta, \quad X_5 = p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s, \\ &X_6 = a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho, \quad X_7 = \sin\omega t\,\partial_a + \omega\cos\omega t\,\partial_u, \quad X_8 = \cos\omega t\,\partial_a - \omega\sin\omega t\,\partial_u, \end{aligned}$$

where $\omega = \sqrt{2\lambda g}$.

We choose ρ and θ are the local coordinates on the quotient, hence we restrict our consideration to a domain, where $\hat{d}\rho \wedge \hat{d}\theta \neq 0$ or, equivalently, $\rho_a\theta_t - \rho_t\theta_a \neq 0$.

There are four relations (syzygies) between the following second order invariants

$$\frac{du_a}{d\rho}, \quad \frac{d\rho_a}{d\rho}, \quad \frac{d\theta_a}{d\rho}, \quad \frac{d(\theta_t + u\theta_a)}{d\rho}, \quad \frac{du_a}{d\theta}, \quad \frac{d\rho_a}{d\theta}, \quad \frac{d\theta_a}{d\theta}, \quad \frac{d(\theta_t + u\theta_a)}{d\theta}, \tag{2}$$

where

$$\frac{d}{d\rho} = \frac{1}{\rho_t\theta_a - \rho_a\theta_t}\left(\theta_a\frac{d}{dt} - \theta_t\frac{d}{da}\right), \qquad \frac{d}{d\theta} = \frac{1}{\rho_t\theta_a - \rho_a\theta_t}\left(-\rho_a\frac{d}{dt} + \rho_t\frac{d}{da}\right)$$

are Tresse derivatives.

Choosing the invariants ρ, θ, v_a, ρ_a, θ_a, $\theta_t + v\theta_a$ as Lie–Tresse coordinates x, y, K, L, M, N respectively, we obtain the quotient equation $\mathcal{E}_q$ as a PDE system for the functions K, L, M, N of (x,y).

With characteristic vector fields of $\mathcal{E}_q$ we obtain following solutions of $\mathcal{E}_q$

$$L = 0,\ M = c_1 x^{1+\frac{2}{n}},\ K = \sqrt{c_2x^2 - \omega^2},\ N = -\frac{2y\sqrt{c_2x^2 - \omega^2}}{n}, \tag{3}$$

$$L = c_1\left(\frac{x}{y}\right)^{\frac{2n}{n-2}},\ M = \frac{y}{x}L,\ K = \sqrt{c_2\left(\frac{x}{y}\right)^{\frac{2n}{n-2}} - \omega^2},\ N = -\frac{2Ky}{n}, \tag{4}$$

where $c_1 \neq 0$, c_2 are constants.

We add solution (3) to the Euler system (1), for the case of ideal gas with n degrees of freedom, and thus obtain a finite-type system of PDEs, which has the following solution

$$\rho = \frac{\omega}{\sqrt{c_2}\cos(c_3 - \omega t)}, \qquad v = a\,\omega\tan(c_3 - \omega t) + f(t),$$

$$\theta = c_1 a\rho^{1+\frac{2}{n}} - \cos^{-\frac{2}{n}}(c_3 - \omega t)\left(c_1 c_2^{-\frac{n+2}{2n}}\omega^{\frac{n+2}{n}}\int\frac{f(t)}{\cos(c_3 - \omega t)}\,dt + c_5\right),$$

where

$$f(t) = \frac{-Rc_1 c_2^{-\frac{n+2}{2n}} \omega^{\frac{n+2}{n}}}{\cos(c_3 - \omega t)} \left(\int \cos^{-\frac{2}{n}}(c_3 - \omega t)\, dt + c_4 \right)$$

and $c_1 \neq 0$, $c_2 > 0, \dots, c_5$ are constants.

Similarly, for the solution (4), we have another solution to (1)

$$u = a\,\omega \tan(c_3 - \omega t) + 2f(t), \quad \theta = \rho \left(\frac{c_2 \cos^2(c_3 - \omega t)}{\omega^2} \right)^{\frac{n-2}{2n}},$$

$$\rho = \frac{c_1 \omega^2 a}{c_2 \cos^2(c_3 - \omega t)} - \frac{c_1 \omega^2}{c_2 \cos(c_3 - \omega t)} \left(\int \frac{2f(t)}{\cos(c_3 - \omega t)}\, dt + c_5 \right),$$

where $f(t)$ is the same as above and $c_1, \dots, c_5$ are constants.

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Stability Analysis of Delay Systems Via a Modified Razumikhin Theorem

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This work is focused on a generalization of the modified Razumikhin theorem with artificially increased delay, as presented in the book by Kolmanovskii and Myshkis, to the case of time-varying differential-difference systems of retarded type. A sufficient conditions for uniform stability of the zero solution is proved under the assumption that standard Picard-type conditions for local extendability of solutions hold. The proof is based on the construction of a suitable Lyapunov–Krasovskii functional using a given Razumikhin function. It turns out that the constructed functional is, in a certain sense, an approximation of a functional satisfying the converse Krasovskii theorem.

Keywords: time delay system, Razumikhin approach, time-varying delay, stability

Consider nonlinear system of differential equations with a bounded delay $h > 0$:

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where x_t is a map from $[-h, 0]$ to $\mathbb{R}^n$, defined by $x_t(\theta) = x(t + \theta)$. It is supplemented with initial condition

$$x(t, t_0, \varphi) = \varphi(t - t_0), \;\; t \in [t_0 - h, t_0],$$

where φ is a continuous function, $t_0 \geq 0$. The space of continuous functions defined on the interval $[-h, 0]$ taking values in $\mathbb{R}^n$ is equipped with the norm

$$\|\varphi\|_h = \sup_{\theta \in [-h,0]} \|\varphi(\theta)\|,$$

where the norm on the right-hand side is the Euclidean vector norm.

For this system, a result on the local extendability of all solutions is known [1]: the right-hand side $f : \mathbb{R} \times \mathbf{C}^{(0)}([-h, 0], \mathbb{R}^n)$ of the system is required to be jointly continuous in two variables and locally Lipschitz in the second argument.

In this paper, we present the following sufficient condition for uniform stability.

Theorem 1. *Assume that for system* (1) *the conditions for local extendability of solutions are satisfied and* $f(t, 0) = 0$, $t \geq 0$. *Assume that there exist positive real numbers* α_1, α_2, *a nonnegative real number* N *and a continuous function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, *such that*

$$\alpha_1 \|x\|^2 \leq V(t, x) \leq \alpha_2 \|x\|^2$$

for any $t \geq 0$, $x \in \mathbb{R}^n$, *and its upper right-hand Dini derivative along solutions of system* (1)

$$\mathcal{D}^+ V(t, x_{t+N}(t, \varphi)) \leq 0$$

for any $t \geq 0$ *and* $\varphi \in \mathbf{C}^{(0)}([-h, 0], \mathbb{R}^n)$, *satisfying condition*

$$V(t + \theta, x(t + \theta, t, \varphi)) \leq V(t + N, x(t + N, t, \varphi)), \quad \theta \in [-h, N].$$

Then the zero solution of system (1) *is uniformly stable.*

For the case of a time-invariant system and a time-independent function V a slightly different version of the theorem was presented in the book [2]. It is worth noting that when $N = 0$ the result reduces to the classical Razumikhin theorem.

We carried out the proof for the general case via the Lyapunov–Krasovskii theorem. We took

$$v(t, \varphi) = \sup_{\theta \in [-h,N]} V(t + \theta, x(t + \theta, t, \varphi)) \tag{2}$$

as the functional. After this, it remains to prove that the functional is continuous, positive definite, admits an infinitesimal upper bound, and its derivative along solutions of system (1) is nonpositive. Positive definiteness is almost obvious, nonpositivity of the derivative is proved similarly to the proof of the classical Razumikhin theorem in [1], the proof of continuity is based on the continuous dependence of solutions on initial functions. The greatest technical difficulty is presented by proving the existence of an infinitesimal upper bound, which is established through the local Lipschitz property of the right-hand side of the system.

It is interesting to note that, for an appropriate choice of the function V in the limit as N tends to infinity, functional (2) coincides with a functional satisfying the converse Krasovskii theorem, which was proposed in [3]. This observation suggests that, as N increases, the sufficient stability condition that we proved also becomes necessary.

It is worth noting that previously in [4] this result, without being strictly formally proved, had already been used to analyze stability of a class of linear systems of the form

$$\dot{x}(t) = Ax(t - \tau(t)),$$

where τ is an admissible bounded delay. This approach made it possible to significantly improve the existing lower bounds of the region of uniform stability. The result presented in this paper justifies the correctness of the constructions carried out therein.

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A Sphere Packing Approach to Multi-Objective Optimization Problem and Its Applications⋆.

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Modelling competitive behaviour in oligopolistic markets through multiobjective optimization is closely linked to decision-making in economic systems. The competitive behaviour of Mongolia, Russia, and Australia in the Chinese coal market is modelled as a multiobjective optimization problem. In this paper, we propose a new approach for identifying Pareto optimal solutions based on sphere packing theory. We introduce a new concept of a set of ϵ-approximate Pareto points. To determine these points, a sphere packing approach is employed. The numerical results indicate that the optimal export quantities of the three countries strongly depend on the price–cost relationship. The results may provide useful insights for designing export strategies in oligopolistic markets.

Keywords: oligopoly market, Pareto points, multiobjective optimization, sphere packing

1 The main results

We formulate the competitive behaviour of three countries, Mongolia, Russia, and Australia, on the Chinese coal market as a multiobjective optimization problem. To solve the problem, we introduce a new notion of a set of -approximate Pareto points, which extends the classical concept of a Pareto optimal solution. The sphere-packing approach can also analytically characterise optimal parameter ranges via the centre and radius of a sphere inscribed in a set of ε- approximate Pareto points. This helps countries to make rational decisions regarding the profitability of a business. The proposed approach was tested numerically using MATLAB for coal data from the sources Sxcoal and the General Administration of Customs of China.

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On the Equilibrium of a Timoshenko Plate with a Crack and a Free Edge

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The equilibrium problem for a Timoshenko plate with a crack, subject to homogeneous Neumann boundary conditions on the plate's outer boundary, is considered. Nonlinear boundary conditions on the crack faces are employed to prevent mutual penetration. The existence of a solution to the problem that is noncoercive because of the conditions on the outer boundary is examined. It is proved that a solution exists provided that the integral conditions imposed on the external forces are satisfied. Solvability is established using a direct sum decomposition of the Sobolev spaces containing the solution functions.

Keywords: plate, crack, noncoercive boundary problem, variational inequality

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Guaranteeing Domain for Instrumental Errors of a Strapdown Inertial Navigation System Moving along the Equator*

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We consider the problem of finding a region in the space of instrumental errors of a strapdown inertial navigation system, which guarantees that navigation error stays within a given error tolerance over a specified finite time interval when the

* The study was conducted under the state assignment of Lomonosov Moscow State University.

system moves along the equator. Analytical expression for the guaranteeing domain is obtained via reduction to the Bulgakov problem of finding extreme pitch angle pertubation, which maximizes position error.

Keywords: strapdown inertial navigation system, error equations, guaranteeing domain of instrumental errors, Bulgakov problem of the accumulation of perturbations

A strapdown inertial navigation system (SINS) is used for autonomously calculating the position, velocity, and orientation of a moving object. Typically SINS consists of three accelerometers measuring specific force, three gyroscopes measuring absolute angular velocity, and a computer in which coordinates, velocity, and orientation of the system are determined via integration of the dynamic motion equations and Poisson's kinematic equation [1]. Since inertial sensor measurements inevitably contain errors, calculated navigation solution differs from the true trajectory. For navigation-class SINS, coordinate determination error must not exceed 1 nautical mile = 1.85 km over a 1-hour motion interval. This gives rise to the problem of finding constraints on the instrumental errors of inertial sensors that ensure the specified navigation accuracy. We will refer to set in the space of the instrumental errors corresponding to such constraints as the guaranteeing domain.

We consider the problem of finding a guaranteeing domain for a simplified SINS moving along the Earth's equator. A simplified SINS consists of two accelelerometers located in the equatorial plane and gyroscope perpendicular to that plane. Let f_1, f_2 denote true specific force components, ω - true angular rate, f_1', f_2', ω' – corresponding accelerometers and gyroscope measurements. Inertial sensor instrumental errors model is given by $f_1' = f_1 + \Delta f_1$, $f_2' = f_2 + \Delta f_2$, $\omega' = \omega + \nu$, where Δf_1, Δf_2, $\nu \in \mathbb{R}$ denote biases of accelerometers and gyroscope. Position and orientation of a SINS are given by longitude λ, polar angle on the Earth's equator, and pitch θ, angle between sensitive axis of the first accelerometer and tangent to the equator. Values λ' and θ' are obtained by integration of the motion equations using inertial sensor measurements f_1', f_2', ω'. Let $\Delta x_1 = R(\lambda' - \lambda)$ be position error, where R is the Earth's radius. We will call $G \subset \mathbb{R}^3$ a guaranteeing domain in the space of instrumental errors if and only if G is the maximal set, such that for every combination of instrumental errors $(\Delta f_1,\, \Delta f_2,\, \nu) \in G$ coordinate determination error satisfies $|\Delta x_1(t)| \le \Delta r_{\max}$, $t \in [0,\, T]$ for every trajectory $\{(\lambda(t),\, \theta(t)) \,|\, \lambda(t) \in PC^1[0, T],\, \theta(t) \in PC[0, T],\, |\dot{\lambda}| \ll \omega_0 = \sqrt{g/R}\}$, where ω_0 denotes the so called Schuller frequency, g – gravity acceleration, T – duration of the motion interval and $\Delta r_{\max}$ – a predefined navigation error tolerance. The problem of finding a guaranteeing domain reduces to the problem of finding extreme trajectories on which position error reaches maximum values:

$$\begin{cases} \Delta \ddot{x}_1 + \omega_0^2 \Delta x_1 = ||\Delta f|| \cos(\theta + \phi) - g\alpha_0(\theta_0,\, \Delta f_1,\, \Delta f_2) - g\nu t, \quad \Delta x_1(0) = 0, \quad \Delta \dot{x}_1(0) = 0\,, \\ |\, \Delta x_1(T_1)\,| \to \max_{\theta(.)}, \quad \theta(t) \in PC[0,\, T_1]\,. \end{cases}$$

First equation of the system governs time propagation of the position error [1]. Term $\alpha(0) = (\Delta f_1 \cos\theta_0 - \Delta f_2 \sin\theta_0)/g$ corresponds to the initial alignment error and ϕ denotes such angle that $\Delta f_1 \cos\theta - \Delta f_2 \sin\theta = ||\,\Delta f\,|| \cos(\theta + \phi)$. This problem is a variant of the Bulgakov's problem of the accumulation of perturbations [3]. The solution can be obtained via Pontryagin's maximum principle. Using this solution we find analytical expression for the guaranteeing domain:

$$\max_{\theta(.)} |\Delta x_1(T)| = \frac{||\Delta f||}{g}\left[2k + 2 - ((-1)^k + 1)\cos\omega_0 T\right] + \left(T - \frac{\sin\omega_0 T}{\omega_0}\right)|\nu| \le \frac{\Delta r_{\max}}{R}, \quad k = \left[\frac{T}{\pi/\omega_0}\right]$$

Figure 1 shows the upper part of the guaranteeing domain for a travel interval of $T = 1\,\text{hr}$ and a navigation error tolerance of one nautical mile, $\Delta r_{\max} = 1.85\,\text{km}$.

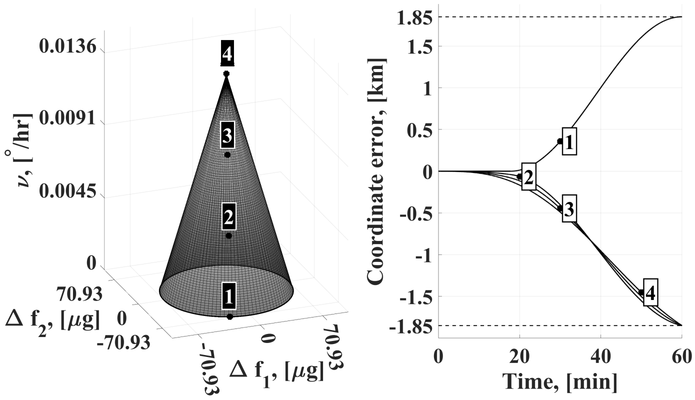

Figure 1: Upper part of the guaranteeing domain and coordinate errors for the extremal trajectories

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Reachable Set Estimates for Impulsive Control Systems with Uncertain Initial State and Cone Constraint on Controls

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The state estimation problem for reachable sets of a linear impulsive control system with initial data uncertainty is considered. The impulsive controls belong to the intersection of a specific cone and a generalized ellipsoid in the space of functions of bounded variation. Ellipsoidal state constraints are imposed on the system. Algorithms for constructing external and internal ellipsoidal estimates of the reachable sets are provided, along with numerical simulation results.

Keywords: impulsive control, reachable sets, state estimation approach

Consider a dynamic control system described by a differential equation with impulsive control $u(\cdot)$:

$$dx(t) = A(t)x(t)dt + du(t), \quad x \in \mathbb{R}^n, \quad t \in [t_0, T], \tag{1}$$

with an unknown but bounded initial state $x(t_0 - 0) = x_0 \in X_0 = E_0 = E(a_0, Q_0)$, where $E(a_0, Q_0)$ denotes an ellipsoid in $\mathbb{R}^n$ with a center a_0 and a symmetric positive definite matrix Q_0. Namely we assume that $A(t)$ is a continuous $n \times n$ – matrix-valued function. The impulsive control $u(\cdot)$ belongs to $\mathbb{V}_p^n$ $(1 \leq p < \infty)$, which denotes the space of n-vector functions of bounded p-variation.

Let $\mathbb{C}_q^n$ denote the space of continuous n-vector functions $y(\cdot)$ with the norm $\|y(\cdot)\|_{\infty,q} = \max_{t_0 \leq t \leq T} \|y(t)\|_q$. In contrast to existing formulations of the problem under study, we impose an additional constraint on impulsive controls: $u(\cdot) \in \mathcal{U} = E^* \cap K^*$, where E^* is a generalized "ellipsoid" in the space $\mathbb{V}_p^n$:

$$E^* = \{u(\cdot) \in \mathbb{V}_p^n | \int_{t_0}^T y(t)^\top du(t) \leq 1, \ \forall y(\cdot) \in \mathbb{C}_q^n, y(t) \in E_0 = E(0, Q_0^{-1}), \ \ \forall t \in [t_0, T]\}$$

and K^* is a generalized cone in the space $\mathbb{V}_p^n$:

$$K^* = \Big\{u(\cdot) \in \mathbb{V}_p^n \mid \int_{t_0}^T y(t)^\top du(t) \geq 0, \quad \forall y(\cdot) \in \mathbb{C}_q^n, \quad y(t) \in K_0, \quad \forall t \in [t_0, T]\Big\},$$

where $K_0 = \{u \in \mathbb{R}^n \mid u = (u_1, \ldots, u_n), \ \ u_1 \geq 0, \ \ u_i \in \mathbb{R}, \ \ i = 2, \ldots, n\}$.

The tube of all possible trajectories of system (1) starting from the initial set $\{t_0, X_0\}$ under the constraints $x_0 \in X_0$, $u(\cdot) \in \mathcal{U}$ is denoted as

$$\mathcal{X}(\cdot) = \mathcal{X}(\cdot; \mathcal{U}, X_0) = \bigcup \Big\{x(\cdot; u(\cdot), x_0) \mid x_0 \in X_0, \ u(\cdot) \in \mathcal{U}\Big\}.$$

Note that the cross-section of the trajectory tube $\mathcal{X}(\cdot)$ at time T coincides with the reachable set $\mathcal{X}(T; \mathcal{U}, X_0)$ of system (1) at time T, starting from the initial set $\{t_0, X_0\}$.

In this study basing on fundamental approaches and results of optimal control and estimation theory [1,2,3,4]. We continue previous researches [5,6,7] and present new state estimation algorithms for a wider class of dynamical control systems under uncertainty.

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Pulse-Sliding Modes in Systems with Aftereffect

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Nonlinear control systems presented in the form of functional-differential inclusions with pulse or discontinuous positional controls are investigated. The formalization of the pulse-sliding mode is carried out. Then the ideal pulse-sliding mode is written. The method of equivalent control for differential inclusion with discontinuous positional controls is used to solve the question of the existence of a discontinuous system for which the ideal pulse-sliding mode is the usual sliding mode. The possibility of the combined use of the pulse-sliding and sliding modes as control actions in those situations when there are not enough control resources for the latter is discussed.

Keywords: functional-differential inclusions, pulse-sliding mode, discontinuous positional controls

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Global Bifurcations of Limit Cycles in a Multi-Parameter Predator-Prey Model

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We study global bifurcations of limit cycles in a special multi-parameter dynamical system of the predator-prey type (the Kohlmeier–Ebenhöh model), which is used to model certain biomedical and ecological relationships associated with intratrophic predation.

Keywords: Kohlmeier–Ebenhöh model, Wintner–Perko termination principle, global bifurcation, limit cycle

1 Introduction

We carry out a global bifurcation analysis of a special multi-parameter dynamical system of the predator-prey type, which is used to model certain biomedical and ecological relationships associated with intratrophic predation. For this purpose, a so-called Kohlmeier–Ebenhöh model and the corresponding cubic-quadratic dynamical system are considered [1], [2].

To control all limit cycle bifurcations in a dynamical system, especially, bifurcations of multiple limit cycles, it is necessary to know the properties and combine the effects of all its rotation parameters. It can be done by means of the development of new bifurcation-geometric methods based on Perko's planar termination principle [3]. This principle is a consequence of the principle of natural termination which was applied for studying one-parameter families of periodic orbits of the restricted three-body problem to show that in the analytic case any one-parameter family of periodic orbits can be uniquely continued through any bifurcation except a period-doubling bifurcation. Such a bifurcation can happen, e. g., in a three-dimensional Lorenz system. But this cannot happen for planar systems. That is why the Wintner–Perko termination principle is applied for studying multiple limit cycle bifurcations of planar polynomial dynamical systems [3]. If we do not know the cyclicity of the termination points, then, applying canonical systems with field rotation parameters, we use geometric properties of the spirals filling the interior and exterior domains of limit cycles.

2 The main results

Consider the Kohlmeier–Ebenhöh model [1]:

$$\begin{aligned} \dot{x} &= x\left(\beta - \varepsilon\, x - \frac{\alpha\, y}{1+(x+\eta\, y)/H}\right), \\ \dot{y} &= y\left(\frac{\gamma + (x+\eta\, y) - \alpha\,\eta\, y}{1+(x+\eta\, y)/H} - \delta\right), \end{aligned} \tag{1}$$

where all parameters, except η, are positive, $\eta \geq 0$ and $\gamma < \alpha$.

In [2], it is shown that (1) can be transformed to a system of the form

$$\dot{x} = y + kxy, \quad \dot{y} = -x + \delta y + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3. \tag{2}$$

System (2) for $k = 0$ is the well-known Kukles system [4]:

$$\dot{x} = y, \quad \dot{y} = -x + \delta y + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3, \tag{3}$$

which was studied in detail in some works; see [2], [5]. However, the qualitative analysis of system (3) remained incomplete for a long time due to the unsolved problem on the maximum number and distribution of its limit cycles [3], [5].

Studying rotation properties of the parameters of (3), we prove the following theorem.

Theorem 1. *System* (3) *with limit cycles can be reduced to the canonical form*

$$\dot{x} = y, \quad \dot{y} = q(x) + (\alpha_0 - \beta + \gamma + \beta\, x + \alpha_2\, x^2)\, y + (c + dx)\, y^2 + \gamma\, y^3, \tag{4}$$

where

1) $q(x) = -x + (1 + 1/a)\, x^2 - (1/a)\, x^3,\ a = \pm 1, \pm 2$ *or*
2) $q(x) = -x + b\, x^3,\ b = 0, -1,$ *or*
3) $q(x) = -x + x^2$;

α_0, α_2, γ *are field rotation parameters and* β *is a semi-rotation parameter.*

Using system (4) and studying global bifurcations of its limit cycles by means of our bifurcation-geometric approach [5], we prove the following theorem.

Theorem 2. *System* (3) *can have at most four limit cycles in* (3 : 1)*-distribution.*

Using Theorem 1 and the Wintner–Perko termination principle which connects the main bifurcations of limit cycles [3], we give also an alternative proof of Theorem 2 [5].

Theorem 3. *There exists no system* (3) *having a swallow-tail bifurcation surface of multiplicity-four limit cycles in its parameter space. In other words, system* (3) *cannot have either a multiplicity-four limit cycle or four limit cycles around a singular point, and the maximum multiplicity or the maximum number of limit cycles surrounding a singular point is equal to three. Moreover, system* (3) *can have at most four limit cycles with their only possible* (3 : 1)*-distribution.*

Finally, applying our bifurcation-geometric approach [5] used to prove Theorems 2 and 3 and taking into account the results of [2], as well as restrictions on the values of the parameters of model (1) related to their biological sense [1], we prove the following theorems.

Theorem 4. System (2) can have at most four limit cycles around a focus located in the first quadrant of the system.

Theorem 5. The Kohlmeier–Ebenhöh model (1) can have at most two limit cycles around a focus in the first quadrant of (1).

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On the Search for Equilibrium in the ORIRES Market Model*

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This report is dedicated to the equilibrium search problem in the ORIRES market model. The company's behavior can be formulated as a linear programming problem. The behavior of certain companies is specific; when modeling such companies, we can develop solution analitically.

Keywords: equilibrium search problem, ORIRES market model, linear programming, energy models

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This report is dedicated to the equilibrium search problem in the ORIRES market model, which was considered in [1]. In the general case, the behavior of company $l \in L$ can be formulated as the following linear programming problem:

$$\sum_{i\in I}\sum_{s\in S}\sum_{t\in T}\tau_s^w(p-c_{li})x_{list}+\sum_{i\in I}\sum_{s\in S}\sum_{t\in T}\tau_s^h(p-c_{li})y_{list}- \tag{1}$$

$$-f\sum_{i\in I}k_{li}(z_{li}-z_{li}^0)-\sum_{i\in I}b_{li}^0 z_{li}\to \max,$$

$$\alpha_{lis}z_{li}\le x_{list}\le \beta_{lis}z_{li},\quad i\in I,\quad s\in S,\quad t\in T, \tag{2}$$

$$\alpha_{lis}z_{li}\le y_{list}\le \beta_{lis}z_{li},\quad i\in I,\quad s\in S,\quad t\in T, \tag{3}$$

$$z_{li}^0\le z_{li}\le \bar{z_{li}},\quad i\in I, \tag{4}$$

$$\tau_s^w\sum_{t\in T}x_{list}+\tau_s^h\sum_{t\in T}y_{list}\le g_s z_{li},\quad i=\text{“HPP”},\quad s\in S, \tag{5}$$

$$\sum_{t\in T}x_{list}\le \eta d z_{li},\quad i=\text{“PSHP”},\quad s\in S, \tag{6}$$

$$\sum_{t\in T}y_{i,l,s,t}\le \eta d z_{li},\quad i=\text{“PSHP”},\quad s\in S \tag{7}$$

where x_{list} and y_{list} – working capacity of station type $i \in I$, belonging to company $l \in L$, in season $s \in S$, in hour $t \in T$, in working days and holidays respectively; z_{li}, z_{li}^0 and $\bar{z}_{li}$ – installed, minimum and maximum allowed capacity of station type $i \in I$, belonging to company $l \in L$, respectively; p – price per unit of electric power; c_{li} and b_{li}^0 – unit costs for electricity generation and unit operating costs of station type $i \in I$, belonging to company $l \in L$, respectively; α_{lis} and β_{lis} – minimum and maximum allowed capacity coefficient of station type $i \in I$, belonging to the company $l \in L$, respectively; τ_s^w and τ_s^h – number of working days and holidays in season $s \in S$ respectively; g_s – maximum number of hours of HPP (Hydroelectric Power Plant) installed capacity used in season $s \in S$; d – maximum number of hours of PSHP (Pumped-Storage Hydropower Plant) installed capacity per day; η – efficiency coefficient of the PSHP.

However, behavior of some companies is specific. There are companies, which don’t plan to develop power in the years immediately ahead, and, due to their geographical location, do not have HPPs or PSHPs. When modeling behavior of such companies, only working power constraints will be presented (2)–(3), using which one can develop a solution analytically. This approach reduces the computational complexity of the equilibrium search problem in the ORIRES market model, which is important in view of high dimensionality of the problem under consideration.

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On the Bounded Solutions of Linear Difference Systems with Periodic Coefficients*

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This paper is devoted to finding bounded solutions to systems of linear difference equations with periodic coefficients under the condition that the spectrum of the monodromy matrix does not intersect the unit circle. Theorems on unique solvability are proved, solution formulas are obtained, and estimates for the norm of the solution are established. In special cases, these estimates coincide with Krein's inequalities.

Keywords: exponential dichotomy, the monodromy matrix, the matricaint, fundamential matrix, the riess projector, discrete Lyapunov equation, Hermitian solution

Consider a system of difference equations

$$x_{n+1} = A(n)x_n + f_n, \quad n \in \mathbb{Z}, \tag{1}$$

where the matrix sequence $\{A(n)\}$ has period N, the matrices $A(n)$ of size $m \times m$ are nondegenerate, and the spectrum of the monodromy matrix $\bar{X}$ does not intersect the unit circle

$$S = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Let $\{X(n)\}$ be the matriciant (fundamental matrix) of the homogeneous system associated with (1) and let

$$P = \frac{1}{2\pi i}\int_S (\lambda I - \bar{X})^{-1}\, d\lambda$$

be the Riess projector.

It follows from [1] that the problem for the system of discrete Lyapunov equations

$$\begin{cases} H(l) - A^*(l)H(l+1)A(l) = ((X(l)^{-*}P^*P(X(l)^{-1})) \\ (X(l))^{-*}((I-P)^*(I-P)(X(l))^{-1}, \quad l = 0, 1, \dots, N-1, \\ H(0) = H(N) > 0, H(0) = (I-P)^*H(0)(I-P) + P^*H(0)P, \end{cases} \tag{2}$$

has a unique Hermitian solution $\{H(l)\}$.

Theorem [2]. For any limited sequence $\{f_n\}$ there exists a unique limited solution $\{x_n\}$ of system (1). This solution can be represented as

$$x_n = \sum_{j=-\infty}^{n-1} X(n)P(X(j+1))^{-1}f_j \; - \sum_{j=n}^{\infty} X(n)(I-P)(X(j+1))^{-1}f_j.$$

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The following estimate holds for the solution:

$$\|x_n\| \leq \left[(1+\sqrt{1-\tfrac{1}{h}}) + (1+\sqrt{1+\tfrac{1}{h}})\right] h\sqrt{M} \sup_{k\in\mathbb{Z}} \|f_k\|, \quad n \in \mathbb{Z},$$

where

$$h = \max_{l=0,\dots,N} \|H(l)\|, \quad M = \max_{l,m=0,\dots,N} \|H^{-1}(l)\| \, \|H(m)\|, \ \{H(l)\}$$

is the solution of (2) and P is the Riess projector.

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The Problem of Selecting the Optimal Pressure at an Oil and Gas Field Using the Katz Method

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This paper addresses the problem of selecting optimal pressure at an oil and gas field using the Katz method. The calculation was performed using Excel software. Based on the results obtained, it can be concluded that for this field, it would be more appropriate to use a 3-stage separation system rather than a 2- or 6-stage system.

Keywords: Katz method, low-viscosity gas-saturated oil, separation stages

1 The main results

Increasing separation efficiency is a critical issue requiring the use of a reasonable number of stages and separation pressure at wellbore collection and processing facilities. Optimizing unit parameters is a pressing issue, as it allows for an increase in oil yield by several percent, which, in today's environment, significantly increases company profits and optimizes the cost of producing one ton of oil.

This paper addresses the problem of selecting the optimal pressure at an oil and gas field using the Katz method [1]. The essence of the method is that the number of moles of oil entering the separator is always equal to the sum of the number of moles of separated gas and separated oil.

The field under study is characterized by low-viscosity gas-saturated oil, which requires several separation stages to most effectively remove associated petroleum gas.

The calculation was performed in Excel using tables and the “Solver” function to find the parameters of the liquid and gas phases.

Graphs were constructed for oil yield versus pressure in a two-stage separation system and for oil yield versus pressure in a three-stage separation system. Based on the results, it can be concluded that a three-stage separation system, rather than a two- or six-stage system, would be more appropriate for this field.

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Infinite-Order Variational Analysis of Optimal Control Problems on Banach Spaces

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The talk is to the optimality criteria in optimal control of nonlinear ordinary differential equations in Banach spaces with an affine dependence on the control. We discuss a strengthening of the usual first-order optimality test. Instead of expanding the cost only along small perturbations, we use an exact formula for the increment of the cost functional. In the smooth case this identity contains the full Taylor expansion and may be understood as an infinite-order variational formula. It leads to a necessary condition close to the feedback minimum principle of V. A. Dykhta. The talk extends this framework to the Banach-space setting, and investigates its relation with Pontryagin’s maximum principle.

Keywords: optimal control, Banach spaces, exact increment formula, Pontryagin’s principle, feedback minimum principle

We address a Mayer’s optimal control problem for a nonlinear ODE in a separable Banach space . The cost has the form

$$\mathcal{J}[u] = \ell(x^u_T),$$

where x^u is the trajectory generated by an admissible control u. The velocity field is assumed to be regular in the state variable and affine with respect to the control. This framework includes the finite-dimensional case, but also covers systems whose natural phase space is infinite-dimensional.

The standard variational scheme starts from small perturbations of a reference control $\bar{u}$. Its first-order part gives Pontryagin’s maximum principle [4]. If the first-order condition is degenerate, one may pass to higher-order conditions, but their explicit use becomes increasingly difficult. The approach discussed in the talk is different: it replaces the hierarchy of separate finite-order tests by one exact nonlinear identity.

Let $\bar{\Phi}$ be the flow generated by $\bar{u}$ and put

$$\bar{p}_t(\mathrm{x}) = \ell(\bar{\Phi}_{t,T}(\mathrm{x})).$$

The function $\bar{p}_t$ is the terminal cost transported backwards along the reference process. For another admissible control u, the increment $\mathcal{J}[u] - \mathcal{J}[\bar{u}]$ can be written through the difference of the corresponding generators applied to $\bar{p}_t$. In the affine case this expression is described by the reference Hamiltonian

$$\bar{H}_t(\mathrm{x}, \mathrm{u}) = D\bar{p}_t(\mathrm{x})[f(\mathrm{x}, \mathrm{u})].$$

The formula is exact and therefore does not discard any higher-order terms.

If the exact formula is restricted to weak variations $\bar{u}+\varepsilon(u-\bar{u})$, the usual Taylor expansion in ε is recovered. Thus, all finite-order variations are already contained in the same identity. This is the sense in which the construction gives an infinite-order variational representation: one works with a single increment formula instead of deriving separate conditions of progressively high orders.

The same formula yields a non-classical optimality criterion. For a fixed reference control $\bar{u}$ one considers comparison controls satisfying

$$\bar{H}_t(x_t^u, u_t) = \min_{\mathrm{u} \in U} \bar{H}_t(x_t^u, \mathrm{u})$$

for a.a. t. The minimization is performed along the trajectory generated by the comparison control itself, not only along the reference trajectory. Hence the construction leads to a fixed-point problem in the control space. If $\bar{u}$ is optimal, no such comparison control may produce a strict descent of the cost. This gives a necessary condition close in spirit to the feedback minimum principle [2].

In Banach spaces, the main analytic points are the existence of a Lipschitz flow, differentiability of the transported terminal cost, and the validity of the generator formula on a suitable class of functions. Once these facts are established, the resulting variational condition has the same form as in the classical finite-dimensional theory. The talk will focus on this passage from the exact increment formula to a new optimality condition, and on its relation to nonlocal improvement methods in optimal control [1,3,5]

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Exact Increment Formulas in Optimal Control of Nonlocal Balance Equations

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This work extends the results of [2] to optimal control problems governed by nonlocal balance equations in the space of compactly supported nonnegative measures on X. We assume that the source is the state measure with a density depending on the current measure and on the control. This class contains nonlocal continuity equations as the zero-source case.
We first obtain an exact increment formula for the nonlocal continuity equation. We then use the barycentric projection from [5] and transfer the formula to the balance setting. This gives, as a byproduct, feedback necessary conditions and a basis for indirect numerical algorithms.

Keywords: nonlocal balance equation, exact increment formula, barycentric projection, feedback necessary condition

Let $\mathcal{M}_c^+(X)$ be the cone of finite nonnegative Borel measures on $X = \mathbb{R}^n$ with compact support, and let $\mu_0 \in \mathcal{M}_c^+(X)$ be chosen. Given a functional $\ell \in C(\mathcal{M}_c^+(X); \mathbb{R})$, and a nonempty compact set $U \subset \mathbb{R}^m$, we denote $\mathcal{U} = \{u \in L^\infty(I; \mathbb{R}^m) : u_t \in U \text{ for a.a. } t \in I\}$ and consider Mayer's optimal control problem on a fixed time period $I = [0, T]$:

$$(P) \qquad \inf\{\mathcal{J}[u] = \ell(\mu_T^u) : u \in \mathcal{U}\}.$$

Here $\mu^u = (\mu_t^u)_{t \in I}$ stands for the weak solution of the balance equation

$$\partial_t \mu_t^u + \operatorname{div}_x(F_t(x, \mu_t^u, u_t)\mu_t^u) = G_t(x, \mu_t^u, u_t)\mu_t^u, \quad \mu_0^u = \mu_0,$$

where $F : I \times X \times \mathcal{M}_c^+(X) \times U \to X$ and $G : I \times X \times \mathcal{M}_c^+(X) \times U \to \mathbb{R}$ have the standard regularity [4]. This type of dynamical systems was studied in [1]. Basic control-theoretical results, including Pontryagin's maximum principle, were obtained in [2] for the μ-linear case, and in [5] for the full statement (P). When $G = 0$, the balance equation reduces to the nonlocal continuity equation, which is naturally considered on the subset $\mathcal{P}_c(\mathbb{R}^n)$ of probability measures.

In the latter case, we obtain an exact increment formula for the functional $\mathcal{J}$: fix $\bar{u} \in \mathcal{U}$, let $\Phi_{s,t}^{\bar{u}} : \mathcal{M}_c^+(X) \to \mathcal{M}_c^+(X)$ be the flow of measures generated by $\bar{u}$, and put $\bar{p}_t(\mu) = \ell(\Phi_{t,T}^{\bar{u}}(\mu))$. We first show that the flow Φ^w, corresponding to a fixed element $w \in U$, admits the a generator in the sense [3]. This generator is the linear unbounded operator

$$\Psi \mapsto \mathcal{L}_t^w \Psi, \quad (\mathcal{L}_t^w \Psi)(\mu) = \int_X \nabla_\mu \Psi(\mu)(x) \cdot F_t(x, \mu, w) \, \mathrm{d}\mu(x),$$

whose domain contains all functionals $\Psi\colon \mathcal{M}_c^+(X) \to \mathbb{R}$ admitting bounded and Lipschitz intrinsic derivative ${}_\mu\Psi(\mu)(x)$ [3]. For every $u \in \mathcal{U}$, we then obtain the following comparison formula:

$$\mathcal{J}[u] - \mathcal{J}[\bar{u}] = \int_0^T \Big((\mathcal{L}_t^{u_t} \bar{p}_t)(\mu_t^u) - (\mathcal{L}_t^{\bar{u}_t} \bar{p}_t)(\mu_t^u) \Big) \mathrm{d}t. \tag{1}$$

We next transfer this formula to the balance equation by means of barycentric projection to the extended space $\widehat{X} = X \times \mathbb{R}_+$: let $\mathcal{P}_c(\widehat{X})$ be the set of compactly supported probability measures on $\widehat{X}$. The barycentric projection is the operator $\mathfrak{B} : \mathcal{P}_c(\widehat{X}) \to \mathcal{M}_c^+(X)$ defined by

$$\langle \mathfrak{B}\nu, \psi \rangle = \int_{\widehat{X}} r\psi(x) \mathrm{d}\nu(x, r), \quad \psi \in C_b(X).$$

Here r is the mass coordinate. Choose $\nu_0 \in \mathcal{P}_c(\widehat{X})$ such that $\mathfrak{B}\nu_0 = \mu_0$. For $t \in I$, $(x, r) \in \widehat{X}$, $\nu \in \mathcal{P}_c(\widehat{X})$ and $w \in U$, define the lifted vector field

$$\hat{F}_t(x, r, \nu, w) = (F_t(x, \mathfrak{B}\nu, w), rG_t(x, \mathfrak{B}\nu, w)) \in \mathbb{R}^{n+1}.$$

If $\nu^u = (\nu_t^u)_{t \in I}$ solves the continuity equation

$$\partial_t \nu_t^u + \mathrm{div}_{x,r}(\hat{F}_t(x, r, \nu_t^u, u_t)\nu_t^u) = 0, \qquad \nu_0^u = \nu_0,$$

then $\mu_t^u = \mathfrak{B}\nu_t^u$ solves the original balance equation. Thus the semilinear source is represented as transport in the additional mass coordinate.

Let $\widehat{\Phi}_{s,t}^{\bar{u}} : \mathcal{P}_c(\widehat{X}) \to \mathcal{P}_c(\widehat{X})$ be the flow of the lifted continuity equation generated by $\bar{u}$, and set $\hat{p}_t(\nu) = \ell((\mathfrak{B}\widehat{\Phi}_{t,T}^{\bar{u}})(\nu))$. For an intrinsically smooth functional Ψ on $\mathcal{P}_c(\widehat{X})$ define

$$(\widehat{\mathcal{L}}_t^w \Psi)(\nu) = \int_{\widehat{X}} {}_\nu\Psi(\nu)(x, r) \cdot \hat{F}_t(x, r, \nu, w) \mathrm{d}\nu(x, r).$$

Applying (1) to the continuity equation with the lifted vector field $\hat{F}$ gives the exact cost-increment formula for the full problem (P):

$$\mathcal{J}[u] - \mathcal{J}[\bar{u}] = \int_0^T \Big((\widehat{\mathcal{L}}_t^{u_t} \hat{p}_t)(\nu_t^u) - (\widehat{\mathcal{L}}_t^{\bar{u}_t} \hat{p}_t)(\nu_t^u) \Big) \mathrm{d}t, \quad \mathfrak{B}\nu_t^u = \mu_t^u.$$

This identity implies a feedback necessary condition: if $\bar{u}$ is optimal, then the right-hand side is nonnegative for every admissible comparison control, in particular for controls obtained by pointwise minimization of $w \mapsto (\widehat{\mathcal{L}}_t^w \hat{p}_t)(\nu_t^u)$. The latter leads to indirect monotone numerical methods of the type [2].

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Methods of Landscape Analysis of Multiextremal Functions and Applications

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The paper considers non-optimization methods of landscape analysis of multiextremal functions, including estimates of the degree of convexity, quadraticity, non-separability, growth constants, the number and ravine character of extrema. Landscape analysis is considered as a tool supporting both the preliminary selection of optimization methods and directly the stages of constructing mathematical models. Within the framework of the classification of methods, pre-optimization analysis is distinguished, aimed at studying the properties of a function over the entire feasible set, and post-optimization analysis, focused on studying the neighborhood of the found solutions.

Keywords: landscape analysis, multiextremal functions, convexity estimation, optimal control

Works on landscape analysis (see, e.g., [1,2,3]) – known in different countries under various names – first appeared in the 1980s. This direction acquired a systematic character in Russia in the early 2000s, and abroad somewhat later. Landscape analysis pursues two main goals. The first is optimization support: obtaining preliminary information that may facilitate the solving of optimization problems and the selection of appropriate methods. The second is modeling support: landscape analysis can serve as a useful tool directly at the stages of model construction, allowing the researcher, through a formalized approach, to obtain information in advance about the types of models being embedded and the potential problems of their optimization.

The paper discusses non-optimization methods for studying multiextremal functions. The main types of landscape analysis include: estimates of the degree of convexity (concavity) of a function, estimates of the degree of quadraticity (non-quadraticity) and quarticity (non-quarticity), estimates of the degree of non-separability, estimates of growth constants from the first to the fourth order inclusive, etc. The classical approach to the analysis of a multiextremal function comprises methods for estimating the number of extrema, estimating the degree of ravine character of extrema, estimating the accuracy of gradients and their approximate values, and others.

As an application, the problem of loss of convexity in optimal control models for a linear system with quadratic functionals is considered. For these problems, a study is carried out using landscape analysis methods and the appearance of large gradients in these problems is demonstrated. The paper presents the results of numerical experiments.

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On Internal Geometric Structure of Convex Sets and Their Faces*

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The talk deals with convex sets that belong to an infinite-dimensional real vector space X with no topology. We endow each convex set $Q \subset X$ with a preorder relation $\trianglelefteq_Q$, called the dominance relation, the definition of which is based on the geometric properties of each particular set Q, and therefore, the dominance relation $\trianglelefteq_Q$ is specific to each Q. The results presented in the talk demonstrate that the view of a convex set Q as a preordered set $(Q, \trianglelefteq_Q)$ can be effectively used in studying the geometric structure of both the convex set Q itself and its faces.

Keywords: convex sets, faces, cones, halfspaces, preorder relations, upper semilattice

1 Introduction

We assume that the convex sets in question belong to an abstract (generally, infinite-dimensional) real vector space X that has no topology. We endow each convex set $Q \subset X$ with a preorder relation $\trianglelefteq_Q$, which is specific to each Q. We refer to the preorder relation $\trianglelefteq_Q$ as the dominance relation on the set Q. In the paper we demonstrate that the definition of the preorder relation $\trianglelefteq_Q$ on a convex set Q is fruitful in studying geometric structure of

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the set Q and its faces. As elementary components of geometric structure of a convex set Q we consider equivalence classes of the quotient set $\mathcal{O}(Q) := Q/\lessgtr_Q$ of the set Q by the equivalence relation $\lessgtr_Q$, where $\lessgtr_Q$ is the symmetric part of the dominance relation $\trianglelefteq_Q$. The family $\mathcal{O}(Q)$ of equivalence classes is a partition of Q, that is $Q = \bigcup\{E \mid E \in \mathcal{O}(Q)\}$, and $E_1 \bigcap E_2 = \emptyset$ for all $E_1, E_2 \in \mathcal{O}(Q), E_1 \neq E_2$. The dominance relation $\trianglelefteq_Q$ induces a partial order relation $\trianglelefteq_Q^*$ on the family $\mathcal{O}(Q)$ of equivalence classes with respect to which the partially ordered family $(\mathcal{O}(Q), \trianglelefteq_Q^*)$ is an upper semilattice. Geometric structure of a convex set Q is completely characterized by the upper semilattice $(\mathcal{O}(Q), \trianglelefteq_Q^*)$.

We study also connections between the geometric structure of a convex set and the geometric structure of its faces.

2 Open components as elementary constituents of convex sets

Let $Q \subset X$ be a convex set. For a pair of points $x, y \in Q$ we say [1] that *a point $x \in Q$ dominates a point $y \in Q$* if there exists a positive real number $\delta > 0$ such that $x - \delta(y - x) \in Q$. The binary relation defined on Q in such way is called [1] *the dominance relation* and is denoted by $\trianglelefteq_Q$. It is worth noting that the dominance relation $\trianglelefteq_Q$ is specific for each convex set Q on which it is defined.

The dominance relation $\trianglelefteq_Q$ is a preorder (i.e., a reflexive and transitive binary relation) on Q. By the symbol $\lessgtr_Q$ we denote the symmetric part of $\trianglelefteq_Q$ defined by $x_1 \lessgtr_Q x_2 \Leftrightarrow x_1 \trianglelefteq_Q x_2, x_2 \trianglelefteq_Q x_1$. The relation $\lessgtr_Q$ is an equivalence relation on Q.

Let $\mathcal{O}(Q) := Q/\lessgtr_Q$ be the quotient set of the convex set Q by the equivalence relation $\lessgtr_Q$. The set Q is the disjunctive union of all equivalence classes from $\mathcal{O}(Q)$. The quotient set $\mathcal{O}(Q)$ is equipped with the partial order $\trianglelefteq_Q^*$ which is defined as follows: $E_1 \trianglelefteq_Q^* E_2$ holds for $E_1, E_2 \in \mathcal{O}(Q)$ if and only if $x_1 \trianglelefteq_Q x_2$ for all (some) $x_1 \in E_1$ and all (some) $x_2 \in E_2$.

Proposition 1. [1] *Every equivalence class E from $\mathcal{O}(Q)$ is relatively algebraically open convex subset of the set Q. Moreover, $E = \mathrm{icr} F_Q(E)$, where $F_Q(E) := \bigcup\{\tilde{E} \in \mathcal{O}(Q) \mid \tilde{E} \trianglelefteq_Q^* E\}$ and $\mathrm{icr} F_Q(E)$ is the relative algebraic interior of a set $F_Q(E)$.*

We refer to subsets E from $\mathcal{O}(Q)$ as open components of Q. The open component containing a point $x \in Q$ is denoted by E_x.

Recall that a partial ordered set $(Z, \preceq)$ is called *an upper semilattice*, if every pair of elements $\{z_1, z_2\}$ from Z has the least upper bound in Z, which is usually denoted by $z_1 \vee z_2$.

Theorem 1. [1] *The family $\mathcal{O}(Q)$ of open components of a convex set Q, ordered by the partial order $\trianglelefteq_Q^*$, is an upper semilattice, and for any $x, y \in Q$ and any number $\alpha \in (0, 1)$ the equality $E_x \vee E_y = E_{\alpha x + (1-\alpha) y}$ holds, in particular, $E_x \vee E_y = E_{\frac{x+y}{2}}$.*

We identify the internal geometric structure of a convex set $Q \subset X$ with the structure of the upper semilattice $(O(Q), \trianglelefteq_Q^*)$.

A nonempty convex subset F of a convex set $Q \subset X$ is called *a face* of Q if $\alpha u + (1-\alpha) v \in F$ for $u, v \in Q$ and $\alpha \in (0, 1)$ implies $u, v \in F$.

The following theorem reveals the connection between the internal geometric structure of a convex set and the internal geometric structure of its faces.

Theorem 2. [1] *Let F be a face of a convex set $Q \subset X$. Then*

(i) any open component $E \in O(Q)$ of the set Q such that $E \bigcap F \neq \emptyset$ belongs to F;

(ii) the dominance relation $\trianglelefteq_F$, defined on the convex set F, coincides with the restriction to F of the dominance relation $\trianglelefteq_Q$ defined on Q;

(iii) every open component $D \in O(F)$ of the face F is an open component of the set Q; i.e., $O(F) \subset O(Q)$.

(iv) the partial order relation $\trianglelefteq_F^*$ defined on $O(F)$ coincides with the restriction to $O(F)$ of the partial order relation $\trianglelefteq_Q^*$ defined on $O(Q)$.

The specific features of the geometric structure of convex cones and halfspaces (convex sets whose complements are convex) are studied in [1,2].

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On Local Search in Mahalanobis-distance-based Clustering Problem

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We address a generalization of the minimum-sum-of-squares clustering problem, where the measure of similarity is given as the squared Mahalanobis distance. In contrast to previous research on k-means algorithms with Mahalanobis distances, we consider such a problem as a continuous non-convex optimization problem and apply the DC (difference of convex) optimization method to solve it. We developed an algorithm for finding local solutions based on solving a sequence of special convex programs that admit of a nearly analytical solution. The results of computational experiments on test instances and the comparison of the proposed approach with the most popular k-means clustering algorithm are performed.

Keywords: supervised machine learning, Mahalanobis distance, local search, k-means, distance metric learning

Clustering with Mahalanobis distance is a generalization of the k-means problem. The problem can be formulated as the following non-convex optimization problem [1]. Given a set $J = \{1, \dots, m\}$ of objects, represented as $a^j \in \mathbb{R}^n$, $j \in J$. The problem is to find k cluster centers $y^i \in \mathrm{R}^n$, $i \in I = \{1, \dots, k\}$, such that the total sum of squared Mahalanobis distances between objects and their closest centers (error in clustering) is minimized:

$$f(x,y) = \sum_{i=1}^{k}\sum_{j=1}^{m} x_{ij}\|y^i - a^j\|_M^2 \downarrow \min_{(x,y)}, \quad x \in S, \quad y \in \mathbb{R}^{k\times n}, \tag{1}$$

where $S = \{x_{ij} \geq 0 : \sum_{i=1}^{k} x_{ij} = 1,\ j = 1, \dots, m\}$, $\|y^i - a^j\|_M^2 = \langle (y^i - a^j), M(y^i - a^j)\rangle$, $M = M^T \succeq 0$. Here x_{ij} is the assignment variable: $x_{ij} = 1$ if data item j is assigned to cluster i, $x_{ij} = 0$ otherwise. Since data items are always assigned to the closest cluster centers, x_{ij} takes binary values for fixed centers y^i in the corresponding optimal solution. Consequently, a natural relaxation with $x_{ij} \geq 0$ has substantiated in the problem (1).

To construct the matrix defining Mahalanobis distance, we solved the following convex optimization problem:

$$\left.\begin{array}{l} \sum\limits_{(a^i,a^j)\in \mathbb{D}} \|a^i - a^j\|_M^2 \to \max\limits_{M}, \\ \sum\limits_{(a^i,a^j)\in \mathbb{S}} \|a^i - a^j\|_M^2 \leq 1, \quad M \succeq 0, \end{array}\right\} \tag{2}$$

where $\mathbb{D}$ and $\mathbb{S}$ are sets of points' pairs known to be from different clusters and from the same cluster, correspondingly. The goal of the problem (2) is to maximize the distance between points from different clusters, and reduce the distance between points from the same cluster. The property of positive semi-definiteness of the matrix $M = \{\mu_{ij}\}$, $M \in \mathbb{R}^{n\times n}$, can be achieved, for instance, by the following constraints: $\sum\limits_{l=1,\ l\neq p}^{n} \mu_{lp} \leq \mu_{pp}$, $p = 1, \dots, n$, which represent the matrix's main diagonal domination. Since the problem (2) is convex one, it can be solved by any conventional convex optimization methods or software.

We represented the objective function of the problem (1) as the difference of two convex functions [2]: $f(x,y) = g(x,y) - h(x,y)$,

$$g(x,y) = \sum_{i=1}^{k}\sum_{j=1}^{m} \left[d_1\|y_i - a_j\|^2 + d_2 x_{ij}^2\right], \tag{3}$$

$$h(x,y) = \sum_{i=1}^{k}\sum_{j=1}^{m} \left[d_1\|y_i - a_j\|^2 + d_2 x_{ij}^2 - x_{ij}\|y_i - a_j\|^2\right], \tag{4}$$

thereby obtained the DC minimization problem. For this problem the special local search was developed [3]. It consists of solving the serial linearized (at the point (x_s, y_s)) problems:

$$g(x,y) - \langle \nabla h(x_s, y_s), (x,y)\rangle \to \min, \quad x \in S,\ y \in \mathbb{R}^{k\times n}. \tag{5}$$

When choosing the DC representation (3)–(4), the linearized problems (5) turn out to be separable by groups of variables, which allows us to develop an effective solution method, the component y in which is calculated analytically:

$$y_i^{(s+1)} = y_i^{(s)} + \Delta y_i^{(s)}, \quad \Delta y_i^{(s)} = \frac{1}{md_1}\sum_{j=1}^{m}(a_j - y_i^{(s)})x_{ij}^{(s)}, \quad i = 1, \dots, k.$$

Next, to update the variable x, it is necessary to solve auxiliary problems ($p \leq k$) for each fixed j:

$$\phi(x) = \sum_{i=1}^{p}\left(d_2 x_i^2 - q_i x_i\right) \to \min, \quad \sum_{i=1}^{p} x_i = 1, \quad x_i \geq 0,\ i = 1, \dots, p,$$

where $q_i = \nabla_{x_{ij}} h(x_s, y_s)$, $q_1 \geq q_2 \geq \dots \geq q_k$. Based on the ideas from [4], two fast algorithms were developed to solve this problem: *Upper* and *Lower*, starting with $p = k$ and $p = 1$, respectively. Computational experiments have shown that the developed method for solving problems (5) works 20 times faster than the universal Clarabel solver for the Python.

Finally, we tested the developed local search method (LSM) and compared it with the k-means algorithm in terms of the quality of the clustering. A computational experiment was carried out on test data sets from the UCI MLR database (https://archive.ics.uci.edu) as well as on the so-called BIRCH test data collection [5]. It showed that the generalization of k-means clustering model can be used to significantly improve clustering performance. The quality of clustering was estimated both by the value of the sum of square-error of all clusters and by using clustering performance metrics: Precision, Recall, and F-measure. The LSM demonstrated a lower clustering error and a better metrics compared with k-means algorithm. Thus, the developed LSM is preferable for accurate clustering.

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Optimal Control Problem for Acoustic Equation and Algoritm for Finding Approximate Solution in One Special Case

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1 The main results

In direct problems of mathematical physics, one strives to find a function that describes various physical phenomena. In this case, the properties of the medium under study are assumed to be known. Then there arise inverse problems in which based on the information about the solution of the direct problem, it is necessary to determine the coefficients of the equations.

In many cases these problems are all-posed. On the other hand, the sought for coefficients of equations are important characteristics of the media being studied. Therefore, the study of such inverse problems of mathematical physics is very important both from theoretical and practical point of view [1–5]. One of the powerful tools for studying inverse problems is the optimization method [1].

Not that in [6-8] some related problems were considered.

1.Problem statement Let us consider the problem of determining a pair of functions from the following relations

$$(u(x,t), \upsilon(x))$$

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} \upsilon_i(x)\frac{\partial u}{\partial x_i} = f(x,t), \ (x,t) \in Q, \tag{1}$$

$$u(x,0) = u_0(x), \ \frac{\partial u(x,0)}{\partial t} = u_1(x), \ x \in \Omega \tag{2}$$

$$\frac{\partial u}{\partial \nu_A}|_S = 0, \tag{3}$$

$$\int_0^T K(x,t)\, u(x,t)\, dt = \varphi(x), \ x \in \Omega, \tag{4}$$

where $\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(\nu, x_i)$ is a conormal derivative, $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega., T > 0$ is a given number, $u(x,t)$ is a acoustic pressure, $\upsilon(x) = (\upsilon_1(x), ..., \upsilon_n(x))$ is a vector function that is expressed by the function of the density of the medium and speed of wave propagation in the medium.

If the functions $\upsilon(x), f(x,t), u_0(x), u_1(x)$ are given, we get direct problem (1)-(3) of determining the function $u(x,t)$. If $\upsilon(x)$ is an unknown function, we will additionally set the condition (4). Then we get the inverse problem (1)-(4) of determining a pair of functions $(u(x,t), \upsilon(x))$.

Assume that we are given the functions $f \in L_2(Q)$, $u_0 \in W_2^1(\Omega)$, $u_1 \in L_2(\Omega)$, $K(x,t) \in L_\infty(Q)$, $\varphi \in L_2(\Omega)$.

We reduce the problem (1)-(4) to the following optimal control problem: find such a vector-function $\upsilon(x)$ from the set

$$V = \left\{\upsilon(x) = (\upsilon_1(x), ..., \upsilon_n(x)), \upsilon_i(x) \in C^1(\bar{\Omega}) : |\upsilon_i(x)| \le M_1, \left|\frac{\partial \upsilon_i(x)}{\partial x_j}\right| \le M_2,\right.$$

$$i,j = 1,...,n \quad almost\, everywhere\, in\ \Omega\}, \tag{5}$$

that affords a minimum to the functional

$$J(\upsilon) = \frac{1}{2}\int_\Omega \int_0^T (K(x,t)\, u(x,t;\upsilon)\, dt - \varphi(x))^2\, dx \tag{6}$$

under constraints (1)-(3), where $u(x,t;\upsilon)$ is the solution of the problem (1)-(3), for $\upsilon = \upsilon(x)$, M_1, $M_2 > 0-$ are the given numbers. We call this problem as $(1)-(3), (5), (6)$.

2.On solvability of the problem (1)–(3),(5),(6)

Theorem 1. Let the conditions accepted when setting the problem $(1)-(3), (5), (6)$ be fulfilled.

Then the set of optimal controls of this problem $V_* = \left\{\upsilon_* \in V : J(\upsilon_*) = \min_{\upsilon \in V} J(\upsilon)\right\}$ is nonempty, weakly compact in $W_2^1(\Omega)$ and any minimizing sequence $\{\upsilon_n\}$ weakly in $W_2^1(\Omega)$converges to the set V_*.

Theorem 2. Let the conditions of theorem 1 be fulfilled and $\alpha > 0$. Then there exists such a dense subset G of the space $W_2^1(\Omega)$ that for any $\omega \in G$ the problem of minimization of the functional $I(\upsilon) = J(\upsilon) + \alpha\|\upsilon - \omega\|^2_{C^1(\Omega)}$ on the set has a unigue solution under conditions (1)-(3).

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Solvability of an Inverse Time-Dependent Source Problem in a Semilinear Parabolic Equation with Robin Boundary Condition

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We consider an inverse time-dependent source problem in a semilinear parabolic equation with Robin boundary condition. We use an additional integral measurement to recover a source parameter. We first establish the uniqueness of the weak solution. Then we prove the existence of the weak solution and derive the error estimates of the approximate solution by Rothe's time-discretization method.

Keywords: time discretization, inverse problem, solvability, source parameter, error estimate

Introduction. We are interested in determining the unknown couple $\{u, k\}$ obeying the following semilinear parabolic equation

$$\partial_t u - \triangle u = k(t)u + f(x, t, u, u), (x, t) \in \Omega \times I \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), x \in \Omega \tag{2}$$

and Robin boundary condition

$$\alpha(u(x,t)) + u(x,t) \cdot \nu = g(x,t), (x,t) \in \Gamma \times I, \tag{3}$$

subject to the additional measurement

$$\int_{\Omega} u(x,t)dx = m(t), t \in I \tag{4}$$

where Ω is a Lipschitz domain in $R^N, N \geq 1$, with $\partial\Omega = \Gamma$ and $I = [0,T], T > 0$ in the time frame. Here f, u_0, α, g and m are given.

This inverse problem has many important applications in heat transfer, diffusion phenomena, thermoelasticity, control theory, population dynamics, nuclear reactor dynamics, medical sciences, biochemistry etc.

Recently, this type of problems have been discussed in many papers. In [1,2], they showed the solvability of weak solution and derived error estimate of approximation solution by employing Rothe's time-discretization method in case the unknown source parameter $k(t)$ is included in semi-linear parabolic equation in the form of time-convolution with the unknown function $u(x,t)$. In [3], they discussed the case where the unknown source parameter $k(t)$ and $\triangle u(x,t)$ are included in time-convolution form.

Purpose. The purpose of the research to prove the solvability of the inverse problem and derive error estimates of the approximate solution by time-discretization method in case the unknown source parameter $k(t)$ and unknown function $u(x,t)$ are represented in multiplicative form in the semilinear parabolic equation.

Results. We denote by $(\cdot,\cdot)$ the standard inner product of $L^2(\Omega)$ and $\|\cdot\|$ its induced norm. When working on the boundary Γ we use a similar notation, namely $(\cdot,\cdot)_\Gamma$ and $\|\cdot\|_\Gamma$. By $\|\cdot\|_X$: we denote the norm of Banach space X.

Theorem 1(Uniqueness). Assume that $u_0 \in L^2(\Omega), g \in C(I, L^2(\Gamma)), m \in C^1(I)$. Moreover, the functions f and α are supposed to be Lipschitz continuous in all variables. Then the problem (2.2)-(2.3) has at most one solution $\{u,k\} \in (C([0,T], L^2(\Omega)) \cap L^2((0,T), H^1(\Omega))) \times L^2(0,T)$. Then $\partial_t u \in L^2((0,T),(H^1(\Omega))^*)$.

Theorem 2(Existence). Let the conditions of Theorem 3.1 be satisfied. Then there exits a weak solution $\{u,k\}$ to (2.2) and (2.3) where $u \in (C([0,T], L^2(\Omega)) \cap L^2((0,T), H^1(\Omega)))$, $\partial_t u \in L^2((0,T),(H^1(\Omega))^*)$ and $k \in L^2(0,T)$.

Theorem 3(Error on coefficient). Let the assumptions of Lemma 4.2 hold. Then there exists a constant $C > 0$ independent of the time step τ that satisfies the following inequality :

$$\max_{t\in I} |\overline{k}_n(t) - k(t)| \leq C\tau$$

Theorem 4(Error on solution). Let the assumptions of Lemma 4.2 hold. Then there exists a constant $C > 0$ independent of the time step τ that satisfies the following inequality :

$$\max_{t\in I} \|U_n(t) - u(t)\|^2 + \int_0^T \|U_n(1) - u(t)\|^2 dt \leq C\tau^2.$$

Conclusion. We first established the uniqueness of the weak solution of an inverse time-dependent source problem in a semilinear parabolic equation with Robin boundary condition. Then we proved the existence of the weak solution and derived the error estimates of the approximate solution by Rothe's time-discretization method. The results of the above study are of great importance for the development of science and technology in various fields, such as engineering, chemistry, biology, etc., as they provide an approximate solution of the inverse problem of coefficients of the semilinear parabolic equations that are often encountered in practice.

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Energy Function of Morse-Smale Diffeomorphisms*

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We prove the existence of energy function for Morse-Smale diffeomorphisms without heteroclinic intersections given on some smooth closed manifolds of dimension four and greater.

Keywords: energy function, Lyapunov smooth function for cascades, Morse-Smale diffeomorphisms

1 The main results

We consider dynamical systems generated by iterations of a diffeomorphism $f : M^n \to M^n$ on a closed smooth manifold M^n of dimension $n \geq 1$. The fundamental theorem of dynamical systems theory states that an arbitrary dynamical system have a global Lyapunov function, that is a continuous function decreasing along all non-chain recurrent trajectories and constant at each chain component. Recall that a diffeomorphism $f : M^n \to M^n$ is called *Morse-Smale* (a MS-diffeomorphism) if its nonwandering set Ω_f is finite and the invariant manifolds of periodic points have only transverse intersections. For Morse-Smale diffeomorphisms, the natural question arises about the existence of a smooth and typical (that is, Morse) Lyapunov function.

A Morse function $\varphi : M^n \to \mathbb{R}$ is called an *energy function* of a diffeomorphism f if it is the global Lyapunov function and the set of its critical points coincides with Ω_f. In [1], Pixton

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proved the existence of energy function for MS-diffeomorphisms on two-dimensional manifolds and constructed an example of a MS-diffeomorphism on the three-dimensional sphere S^3 for which the energy function does not exist. This effect is primarily related to the possibility of wild embedding of closures of separatrices of saddle periodic points of the diffeomorphism. In [2], [3], Grines, Laudenbach, and Pochinka obtained necessary and sufficient conditions for the existence of an energy function for MS-diffeomorphisms on three-dimensional manifolds. In particular, it was shown in [3] that if all one-dimensional separatrices of a MS-diffeomorphism $f : M^3 \to M^3$ form trivial frames, then f has the energy function. In [4] it was shown that for $n \neq 4$, the one-dimensional separatrices that do not participate in heteroclinic intersections always form trivial frames. However, a direct generalization of the result and technique of [3] to the case of higher dimensions is impossible. Nevertheless, it is possible to determine a class of diffeomorphisms for which the energy function exists.

Theorem. *Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism without heteroclinic intersections, and M^n be either the sphere S^n or the direct product $S^{n-1} \times S^1$, $n \geq 4$. Then there exists an energy function for f.*

For the case $M^n = S^n$, Theorem follows from [5], where it was proved that for any MS-diffeomorphism $f : S^n \to S^n$ without heteroclinic intersections there exists $m > 0$ such that f^m is topologically conjugated with a gradient flow of a Morse function. In the case when $M^n = S^{n-1} \times S^1$, there exist MS-diffeomorphisms without heteroclinic intersections that are not embedded even in topological flows. Nevertheless, it is possible to construct an energy function for these diffeomorphisms as well.

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Weak Solvability of the Stress-Strain State Problem for a Composite Layered Medium

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A composite elastic medium is considered as a collection of a finite number of layered components (layers), whose interfaces form parts of the boundaries of

adjacent layers. The functions describing the elastic properties of the composite medium belong to the Sobolev space of bounded, square-summable functions with a generalized first-order derivative. A solvability analysis of the boundary value problem describing the stress-strain state of such a composite layered medium is presented in a weak formulation.

Keywords: layered medium, boundary value problem, deformation optimization

A layered domain $\mathfrak{I} \subset \mathbb{R}^3$ with the boundary $\partial\mathfrak{I}$ is defined as a set of bounded subdomains (layers) $\mathfrak{I}_j$, $j = 0, 1, 2, \dots, N$ (with boundaries $\partial\mathfrak{I}_j$), and a set of interfaces between adjacent layers $S_j \subset \partial\mathfrak{I}_j$, $j = 1, 2, \dots, N$ [1].

Let $u = \{u_1, u_2, u_3\} \in \left(W_2^1(\mathfrak{I})\right)^3$ be a vector-valued function that describes the displacements of points in the elastic medium. Let $\sigma = \{\sigma_{il}\}$, $i, l = 1, 2, 3$, $\sigma \in L_2(\mathfrak{I})$, denote the stress tensor, which is related to the displacement function u by Hooke's law [2]:

$$\sigma_{il}(u) = \sum_{k,m=1}^{3} a_{ilkm}(x)\varepsilon_{km}(u),$$

where $\varepsilon_{km}(u)$ are the components of the strain tensor defined as:

$$\varepsilon_{km}(u) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k}\right), \quad k, m = 1, 2, 3.$$

Let the following conditions hold for the elasticity coefficients:

1. $a_{ilkm}(x) \in L_\infty(\mathfrak{I})$,
2. $a_{ilkm}(x) = a_{likm}(x) = a_{ilmk}(x) = a_{kmil}(x)$,
3. $\sum_{i,l,k,m=1}^{3} a_{ilkm}(x)\xi_{il}\xi_{km} \geq C_0 \sum_{i,l=1}^{3} \xi_{il}^2 \quad \forall \xi_{il} = \xi_{li}$, where $C_0 > 0$ is a constant.

Let us define the stress-strain state problem of the composite layered medium in a weak formulation. For any functions $u = \{u_1, u_2, u_3\}$ and $v = \{v_1, v_2, v_3\}$ belonging to $\left(W_2^1(\mathfrak{I})\right)^3$, we introduce the bilinear form:

$$a(u, v) = \int_{\mathfrak{I}} \sum_{i,l,k,m=1}^{3} a_{ilkm}(x)\varepsilon_{km}(u)\varepsilon_{il}(v)\, dx.$$

We also introduce the vector-valued functions $f(x) := (f_1(x), f_2(x), f_3(x)) \in (L_2(\mathfrak{I}))^3$ and $F_j(x) := (F_1^j(x), F_2^j(x), F_3^j(x)) \in (L_2(\mathfrak{I}))^3$, $j = 1, \dots, N$, to define a linear functional on $\left(W_2^1(\mathfrak{I})\right)^3$ as follows:

$$(\mathfrak{L}, v) = \int_{\mathfrak{I}} \sum_{i=1}^{3} f_i(x) v_i\, dx + \sum_{j=1}^{N} \int_{S_j} \sum_{i=1}^{3} F_i^j(x) v_i\, dx.$$

The Weak Formulation of the Stress-Strain State Problem. Find a displacement vector-valued function $u(x)$ such that

$$a(u, v) = (\mathfrak{L}, v) \tag{1}$$

Theorem. *Let the aforementioned assumptions on the coefficients $a_{ilkm}(x)$ and the stress tensor $\sigma_{il}(u)(x)$ be satisfied. Then, problem (1) possesses a unique weak solution $u \in V_0^1$, and the following a priory estimate holds:*

$$\|u\|_{V_0^1}^2 \leq c,$$

where $c > 0$ is a constant independent of u.

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A Difference Scheme for the Stationary Navier–Stokes–Oseen System in the Class of Summable Functions Supported on a Network-like Domain

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A stationary hydrodynamic problem for viscous incompressible fluids is considered, providing a mathematical description of viscous fluid transport through pipelines or main networks. The primary analytical tool is the finite difference method.

Keywords: Network-like domain, boundary value problem, difference scheme

1 Statement of the Problem

A connected network-like domain $\mathfrak{I} \subset \mathbb{R}^3$ with the boundary $\partial\mathfrak{I}$ is defined as a union of a finite set of bounded subdomains (layers or branches) $\mathfrak{I}_j$ (with boundaries $\partial\mathfrak{I}_j$), $j = 1, 2, \dots, N$, and a set of bilateral interfaces S_j ($S_j \subset \partial\mathfrak{I}_j$, where S_j^+ and S_j^- denote the respective sides of the interface S_j), $j = 1, 2, \dots, N-1$.

For the velocity vector-valued function $(x) = (u_1(x), u_2(x), u_3(x))$ and the pressure scalar function $p(x)$ at $x = (x_1, x_2, x_3) \in \mathfrak{I}$, we consider the stationary Navier–Stokes–Oseen system:

$$-\nu\Delta + \sum_{i=1}^{3} a_i(x)\frac{\partial}{\partial x_i} + \operatorname{grad} p = , \quad \operatorname{div} = 0, \quad (x)\big|_{x\in\partial\mathfrak{I}} = 0, \tag{1}$$

subject to the transmission conditions on the internal interfaces:

$$(x)\big|_{S_j^+} = (x)\big|_{S_j^-}, \quad \frac{\partial(x)}{\partial n_j^+} + \frac{\partial(x)}{\partial n_j^-} = 0, \quad j = 1, 2, \dots, N-1. \tag{2}$$

2 The Main Results

The weak solvability of problem (1)–(2) is established using the finite difference method.

Let us define a standard spatial grid $(\mathfrak{I})_h = \bigcup_{j=1}^{N} (\mathfrak{I})_{j\,h}$ over the network-like domain $\mathfrak{I} = \bigcup_{j=1}^{N} \mathfrak{I}_j$, and a corresponding grid $(S)_h = \bigcup_{j=1}^{N-1} (S)_{j\,h}$ over the interfaces $S = \bigcup_{j=1}^{N-1} S_j$, where $(\Im)_{j\,h}$ and $(S)_{j\,h}$ are the discrete grids of the subdomains $\Im_j$ and interfaces S_j for each fixed j.

The corresponding difference scheme is determined by the following system of discrete equations:

$$\begin{aligned} &\nu \sum_{i=1}^{3} {}_{x_i \bar{x}_i} + \sum_{i=1}^{3} a_{i\,h x_i} + \mathrm{grad}_h\, p_h = {}_h, \quad \mathrm{div}_h\, {}_h\big|_{(\Im)_h} = 0, \\ &{}_h\big|_{S_j^+(mh)} = {}_h\big|_{S_j^-(mh)}, \quad \Big({}_{h,n_j^-} + {}_{h,n_j^+}\Big)\big|_{(S)_j(mh)} = 0, \quad {}_h\big|_{\partial \Im_h} = 0. \end{aligned} \tag{3}$$

For the numerical analysis of scheme (3), we utilize the discrete analogue of the weak (integral) identity:

$$\nu \hbar \sum_{(\Im)_h} \sum_{i=1}^{3} {}_{x_i} \eta_{x_i} + \hbar \sum_{(\Im)_h} \sum_{i=1}^{3} a_{i\,h x_i} \eta_h = \hbar \sum_{(\Im)_h} {}_h \eta_h, \tag{4}$$

where $\hbar = h_1 \cdot h_2 \cdot h_3$ is the volume elements of the grid, and η_h is an arbitrary discrete test function satisfying the corresponding boundary and transmission conditions.

Theorem. *Let (x) be a bounded vector-valued function in $(\tilde{W}_0^1(\Im))^3$ and $(x) \in (L_2(\Im))^3$. Then the difference scheme (3) is uniquely solvable and stable with respect to the grid norm of the discrete analogue of the space $(\tilde{W}_0^1(\Im))^3$. Moreover, the multilinear interpolations $\hat{}_h(x)$ of the grid solutions strongly converge to the exact solution (x) in the space $(\tilde{W}_0^1(\Im))^3$ as $|h| \to 0$.*

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Duality of Convex Sets and Smoothness of the Boundary of the Reachable Set with Control Constraints in L_p

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The paper investigates properties of reachable sets of linear control systems with control constraints given by a ball in L_p. It is proved that the boundary of the reachabile set is smooth for $p > 1$. The proof is based on duality properties of convex sets in $\mathbb{R}^n$. An algorithm for simultaneous construction of the reachable set and its polar set is proposed.

Keywords: control system, Integral constraints, Reachable set, Smoothness of the boundary

1 Introduction

In this study we investigate the smoothness of the boundary of the reachable set of a linear nonstationary control system under L_p-constraints with $p > 1$. In [1] assuming analyticity of the system coefficients, it was shown that for $1 < p \leq 2$ the boundary of the reachable set is C^1-smooth. Here we extend the previous result in two directions. First, we admit systems with discontinuous coefficients. Secondly, under these conditions, it is established that the reachable set boundary is a C^1-smooth submanifold of $\mathbb{R}^n$ for any $p > 1$. Compared with [1], the structure of the proof has been substantially revised. Now it relies on the concept of the polar set in $\mathbb{R}^n$ and uses several results from duality theory of convex sets.

2 Support functions and polars of reachable sets

The support function of a set $X \subset \mathbb{R}^n$ is defined by $\psi_X(y) = \sup\{(x, y) : x \in X\}$, $y \in \mathbb{R}^n$, where (x, y) denotes the Euclidean inner product in $\mathbb{R}^n$. The support function enjoys the following well-known characterization.

Proposition 1. *Let X be a compact strictly convex set such that $0 \in intX$. Then:*

1. *For each $y \neq 0$, the supremum of (x, y) on X is attained at a unique point $\hat{x}(y) \in \partial X$, referred to as the support point.*
2. *The map $\hat{x} : \mathbb{R}^n \ \{0\} \to \partial X$ is continuous.*
3. *The function ψ_X is continuously differentiable for $y \neq 0$, with gradient $\nabla\psi_X(y) = \hat{x}(y)$.*

The polar of a set $X \subset \mathbb{R}^n$ is defined by

$$X^\circ = \{y \in \mathbb{R}^n : \ (x, y) \leq 1, \ \forall x \in X\}.$$

The polar is a closed, convex set containing the origin. By definition, $X^\circ = \{y \in \mathbb{R}^n : \ \psi_X(y) \leq 1\}$ so X° is a sublevel set of the support function of X.

Consider the time-varying linear control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t_0 \leq t \leq t_1, \quad x(t_0) = x^0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control; the matrix functions $A(t)$ and $B(t)$ are Lebesgue measurable and bounded on $[t_0, t_1]$. Admissible controls are functions $u \in L_p[t_0, t_1]$ with $p > 1$. For each $u \in L_p[t_0, t_1]$, the product of the functions $B(t)$ and $u(t)$ belongs to $L_1[t_0, t_1]$, and hence (1) admits a unique absolutely continuous solution $x(t, u)$.

Controls are constrained by $u \in B_p(0, \mu)$, where $B_p(0, \mu) \subset L_p$ denotes the closed ball of radius μ centered at the origin.

The reachable set at time t_1 is the collection of terminal states generated by admissible controls: $G = \{x(t_1, u) : \ u \in B_p(0, \mu)\}$. In what follows we assume $x(t_0) = x^0 = 0$ and $\mu = 1$, which entails no loss of generality. Also we assume that control system (1) is completely controllable.

Lemma 1. *The sets G and G° are compact, strictly convex and centrally symmetric bodies in $\mathbb{R}^n$, that contain the origin in their interior. The boundaries of G, G° are given as follows*

$$\partial G^\circ = \{y \in \mathbb{R}^n : \psi_G(y) = 1\}, \quad \partial G = \{x \in \mathbb{R}^n : \psi_{G^\circ}(x) = 1\}.$$

Let us choose an arbitrary direction $s \in S^{n-1} = \{s \in \mathbb{R}^n : \|s\| = 1\}$ and find the point $\bar{y}(s)$ where the ray $\{\lambda s : \lambda \in \mathbb{R}^+\}$ intersects the boundary of the polar reachable set G°. Since $\partial G^\circ = \{s \in \mathbb{R}^n : \psi_G(s) = 1\}$, we get $\bar{y}(s) = \psi_G(s)^{-1}s$.

Lemma 2. *The point $\bar{y}(s)$ is a support point of the set G° in the direction $\hat{x}(s)$. The following equalities hold*

$$\partial G = \{\hat{x}(s) : s \in S^{n-1}\}, \quad \partial G^\circ = \{\bar{y}(s) : s \in S^{n-1}\}.$$

The function $\psi_{G^\circ}(x)$ is continuously differentiable, and its gradient is nonzero on ∂G. The points $\bar{y}(s)$ and $\hat{x}(s)$ are related by the following relations

$$\nabla\psi_G(\bar{y}(s)) = \hat{x}(s), \quad \nabla\psi_{G^\circ}(\hat{x}(s)) = \bar{y}(s), \quad \forall s \in S^{n-1}.$$

Using Proposition 1, Lemmas 1 and 2, and the classical bipolar theorem [2], we obtain the following theorem [3]

Theorem 1. *If system (1) is completely controllable, then the boundary of the reachable set G is an $(n-1)$-dimensional C^1-smooth submanifold of $\mathbb{R}^n$.*

We discuss some indicators of smoothness of the boundary of the reachable set and its polar. A description of an algorithm for simultaneous construction of the reachable set and its polar set is given, the results of numerical simulation are presented.

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The Riman Method for a Second-order Hyperbolic Equation

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In this paper, we consider the Cauchy problem for a nonhomogeneous second-order hyperbolic equation with an exponential coefficient. An explicit form of the Riemann function is obtained. Using the Riemann function method, the solution of the Cauchy problem is studied, and an explicit representation of the solution is derived.

Keywords: Cauchy problem, hyperbolic equation, Goursat problem, Riemann function, Riemann method

1 Introduction

Let a $u(x,y)$ function, $\{(x,y): -\infty < x < +\infty,\ 0 \leq y < +\infty\}$ twice continuously differentiable in a domain, be a solution of the following Cauchy problem:

$$\frac{\partial^2 u}{\partial x^2} - q_1(x)\,u = \frac{\partial^2 u}{\partial y^2} - q_2(y)\,u + p(x,y)\,, \tag{1}$$

$$u(x,0) = f(x)\,,\ u_y(x,0) = g(x)\,, x \in (-\infty,+\infty) \tag{2}$$

where $f(x)$ and $g(x)$ are continuously differentiable functions, and the functions $p(x,y)$, $q_1(x)$, and $q_2(y)$ are continuous in the domains $\{(x,y): -\infty < x < +\infty,\ 0 \leq y < +\infty\}$, $(-\infty,+\infty)$,$(0,+\infty)$ respectively.

Problems of the type (1)-(2) have been studied in detail in the works of many authors (see [1]–[3] and the references therein). On the other hand, it is well known that the Riemann method is effectively applied to Cauchy and Goursat problems for second-order hyperbolic differential equations (see [1]–[4]).

Note that the value of the function $u(x,y)$ at the point (x_0,y_0) can be regarded as the value of a certain linear functional $T_{x_0}^{y_0}$ acting on the vector function $(f(x)\,,g(x))$:

$$u(x_0,y_0) = T_{x_0}^{y_0}(f(x)\,,g(x))$$

The general form of this functional was obtained by B. Riemann (see [4]).

Now consider the problem

$$\frac{\partial^2 u}{\partial x^2} - e^{2x}u = \frac{\partial^2 u}{\partial y^2} - e^{2y}u + p(x,y)\,, \tag{3}$$

$$u(x,0) = f(x)\,,\ u_y(x,0) = g(x)\,, x \in (-\infty,+\infty) \tag{4}$$

Let D be a domain bounded by the characteristics $x - x_0 = \pm(y - y_0)$ Denote by $R(x,y,x_0,y_0)$ a twice continuously differentiable solution of the equation

$$R_{xx} - e^{2x}R = R_{yy} - e^{2y}R \tag{5}$$

in the domain D, which is equal to unity on the characteristics $x - x_0 = \pm(y - y_0)$ the function $R(x,y,x_0,y_0)$ is called the Riemann function.

In the present paper, an explicit form of the Riemann function corresponding to equation (3) is obtained. Consequently, an explicit formula for the solution of problem (3)–(4) is derived.

Theorem 1. The Riemann function $R = R(x,y,x_0,y_0)$ of equation (3) has the form

$$R(x,y,x_0,y_0) = J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}$$

where $z = \sqrt{2\left(e^{x_0+y_0} - e^{x+y}\right)\left(ch(y-x) - ch(y_0-x_0)\right)}$, $(x,y) \in D$and $J_0(z)$ is the Bessel function of the first kind.

Sketch of the proof. Introduce the change of variables $\frac{x+y}{2} = \xi$, $\frac{x-y}{2} = \eta$ and consider the function $\frac{x_0+y_0}{2} = \xi$, $\frac{x_0-y_0}{2} = \eta_0$, where

$$u = \sqrt{2\left(e^{2\xi_0} - e^{2\xi}\right)\left(ch2\eta - ch2\eta_0\right)},\ \eta_0 \leq \eta \leq \xi \leq \xi_0$$

It is easy to verify that

$$\frac{\partial R}{\partial \xi} = 2e^{2\xi}\left(ch2\eta_0 - ch2\eta\right)J_0'(u)\,u^{-1},$$

$$\frac{\partial^2 R}{\partial\xi\partial\eta} = -2e^{2\xi} sh2\eta J_0''(u) - 2e^{2\xi} sh2\eta J_0'(u)\, u^{-1}.$$

Hence it follows that

$$\frac{\partial^2 R}{\partial\xi\partial\eta} - 2e^{2\xi} sh2\eta\, R = -2e^{2\xi} sh2\eta \left(J_0''(u) + J_0'(u)\, u^{-1} + J_0(u)\right) = 0$$

The latter equality implies the validity of relation (5).
Using this theorem, the following result is established.
Theorem 2. The solution of problem (1)–(2) is given by the formula

$$u(x,y) = \frac{f(x+y) + f(x-y)}{2} + \frac{1}{2}\int_{x-y}^{x+y} [g(z)\, R(z.0, x_0, y_0) -$$

$$-f(z)\, R_t(z.0, x, y)]dz + \frac{1}{2}\iint_D R(z.t, x, y)\, p(z,t)\, dzdt$$

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Application of the Hankel Transform to Solving Cauchy Problem for Model Partial Differential Equation

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In this paper, we study the Cauchy problem for a model degenerate partial differential equation involving the Bessel-type differential operator. In particular, we consider a degenerate parabolic equation in appropriate domains. The solution is constructed using the method of separation of variables and the Hankel transform.

Keywords: Hankel transform, Cauchy problem, degenerate equation, Bessel function, Dirac's delta function, Gaussian hypergeometric function

1 Cauchy problem for a model degenerate parabolic equation.

We consider in the domain $\Omega = \{(x,t):\ x>0,\ t>0\}$ degenerate parabolic equation

$$u_t(x,t) = u_{xx}(x,t) + \frac{1}{x}u_x(x,t) - \frac{\nu^2}{x^2}u(x,t),\ |\nu| < \frac{1}{2}. \tag{1}$$

Find a restricted function $u(x,t) \in C^{2,1}(\Omega)$, satisfying the equation (1) and initial condition

$$u(x,t)|_{t=0} = \varphi(x),\ \ x>0, \tag{2}$$

where $\varphi(x) \in C^2$, $\int\limits_0^\infty |\varphi(x)|\sqrt{x}dx < c = const.$

An explicit solution to the problem is constructed using the method of separation of variables in combination with the Hankel transform [3]. The Dirac delta function is used to represent the initial data and to derive integral representations of the solutions [1], [2]. The solution is obtained in terms of modified Bessel functions. We obtain the following solution

$$u(x,t) = \frac{1}{4t}\int\limits_0^\infty \varphi(\rho)\rho e^{-\frac{\rho^2+x^2}{4t}}\left[I_{-\nu}\left(\frac{\rho x}{2t}\right) + I_\nu\left(\frac{\rho x}{2t}\right)\right]d\rho. \tag{3}$$

where $I_\nu(x)$ - modified Bessel function of the first kind and defined [4] as follows

$$I_\nu(x) = \sum_{n=0}^\infty \frac{1}{\Gamma(n+\nu+1)\,n!}\left(\frac{x}{2}\right)^{2n+\nu}.$$

The results obtained generalize classical solutions and demonstrate the effectiveness of the Hankel transform method for degenerate equations with singular coefficients.

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Existence and Uniqueness Results for the Cauchy Problem Involving Regularized Prabhakar Fractional Derivatives

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In this short note, we want to give the main parts of the verification of the ordered Prabhakar fractional order differential operator and the Cauchy problem for an equation with singular coefficients. That is, we present the exact solution of the Cauchy problem for the Prabhakar fractional derivative partial differential equation.

Keywords: Prabhakar fractional derivative operator, Mittag-Leffler type functions, Bessel function, Hankel transform

1 The main results

It is well-known that many special functions appear in solutions for differential equations. For instance, hypergeometric functions are key part of solutions for singular elliptic equations and many other degenerate partial differential equations. In fractional calculus, so called Mittag-Leffler type functions play crucial role. Multivariable analogs of such functions are also important and they are linked with multi-term fractional differential equations [7].

Using the Fourier method, we find the general solution of the given equation. The unknown coefficients are found using the Hankel transformation. We have also presented important statements the bivariate Mittag-Leffler function $E_2(x,\ y)$ [2–3] and the Bessel function. The main objective is the Cauchy problem for a singular coefficient equation involving ordered Prabhakar derivatives.

Let us consider the following time-fractional diffusion equation

$$ {}^{PC}\mathtt{D}_{0t}^{\alpha,\beta,\gamma,\delta} u(x,t) - u_{xx}(x,t) - \frac{\nu}{x} u_x(x,t) = 0, \quad 0<\nu<1, \tag{1}$$

in a domain $\Omega = \{(x,t):\ x>0,\ 0<t<\infty\}$. Here $\alpha,\beta\in \mathrm{R}^+,\ \ \gamma,\delta\in\mathrm{R},\ \ m=[\beta]+1$, $m-1\le\beta<m$ and

$$ {}^{PC}\mathtt{D}_{0t}^{\alpha,\beta,\gamma,\delta} y(t) = {}^{P}I_{0t}^{\alpha,m-\beta,-\gamma,\delta} \frac{d^m}{dt^m} y(t), \tag{2}$$

represents regularized Prabhakar fractional derivative [4] and

$$ {}^{P}I_{0t}^{\alpha,\beta,\gamma,\delta} y(t) = \int_0^t (t-\xi)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left[\delta(t-\xi)^{\alpha}\right] y(\xi)\,d\xi, \ \ t>0, \tag{3}$$

represents Prabhakar fractional integral [6]. We formulate the Cauchy problem for time-fractional diffusion equation in the case of $1<\beta<2$.

It is required to find in the domain a solution to equation (1) satisfying the initial condition

$$ u(x,t)|_{t=0} = \psi(x), \quad u_t(x,t)|_{t=0} = \varphi(x), \quad 0<x<\infty, \tag{4}$$

and for any fixed t we have

$$ \lim_{x\to+\infty} u(x,t) = 0, \quad \lim_{x\to+0} u(x,t) = 0, \tag{5}$$

where $\varphi(x),\ \psi(x)\in\mathbf{C}^2$, $\displaystyle\int_0^\infty |\varphi(x)|\,x^{\frac{\nu}{2}}dx < c = const$ and $\displaystyle\int_0^\infty |\psi(x)|\,x^{\frac{\nu}{2}}dx < c = const$, in addition to this

$$ \varphi(0) = \psi(0) = 0, \quad \lim_{x\to\infty}\varphi(x) = 0, \quad \lim_{x\to\infty}\psi(x) = 0. \tag{6}$$

Solution. We look for the solution of equation (1) using the Fourier method.From the initial conditions (4), we find the unknown functions $C_1(\lambda)$ and $C_2(\lambda)$ using the Hankel substitution. Then the solution of Cauchy problem is as follows

$$u(x,t) = \frac{1}{2} x^{\frac{1-\nu}{2}} \int_0^{\infty} \rho^{\frac{\nu-1}{2}} (\psi(\rho) G_1(x,t,\rho) + \varphi(\rho) G_2(x,t,\rho))\rho d\rho, \tag{7}$$

where

$$G_1(x,t,\rho) = \int_0^{\infty} J_{\frac{1-\nu}{2}}(\lambda\rho) J_{\frac{1-\nu}{2}}(\lambda x) \left(1 - \lambda^2 t^\beta \Gamma(\gamma) E_2 \left(\begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta+1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda^2 t^\beta \\ \delta t^\alpha \end{matrix} \right)\right) \lambda d\lambda,$$

$$G_2(x,t,\rho) = \int_0^{\infty} J_{\frac{1-\nu}{2}}(\lambda\rho) J_{\frac{1-\nu}{2}}(\lambda x)\, t \left(1 - \lambda^2 t^\beta \Gamma(\gamma) E_2 \left(\begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta+2, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda^2 t^\beta \\ \delta t^\alpha \end{matrix} \right)\right) \lambda d\lambda.$$

Here $E_2(\cdot)$ is a bi-variate Mittag-Leffler function [5]

$$E_2 \left(\begin{matrix} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2; \\ \delta_1, \alpha_3, \beta_2; \delta_2, \alpha_4; \delta_3, \beta_3; \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m + \beta_1 n} (\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1 + \alpha_3 m + \beta_2 n)} \frac{x^m}{\Gamma(\delta_2 + \alpha_4 m)} \frac{y^n}{\Gamma(\delta_3 + \beta_3 n)}.$$

Now we show that function (7) satisfies condition (5). For any fixed t, function $T(t)$ is bounded. Further, taking into account the asymptotic representation of the Bessel functions [1] for $z \to \infty$, with $|\arg(z)| < \pi$, then $X \sim c(\lambda) x^{-\frac{\nu}{2}}$. And this means that on the basis of Fourier method we are convinced that the condition (5) is satisfied.

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On Properties of a Sequence of Lyapunov Functions Defined by One Class of High-Dimensional Systems of Differential Equations*

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The report considers a delay differential equation and a sequence of high-dimensional systems of ordinary differential equations. The limit of the sequence of Lyapunov functions is studied, and estimates are proved for the rate of convergence to zero as $t \to \infty$ for the solutions of the problem for the delay differential equation.

Keywords: delay differential equation, high-dimensional system of ordinary differential equations, stability, Lyapunov equation

1 The main results

Consider the following initial value problem for a delay equation:

$$\begin{cases} \dfrac{dy(t)}{dt} = ay(t) + by(t-1), & t > 1, \\ y(t) = \varphi(t), & t \in [0,1], \\ y(1+) = \varphi(1), \end{cases} \tag{1}$$

where $\varphi \in C([0,1])$, $a, b \in \mathbb{R}$. It follows from [1] that the solutions of problem (1) are related to the solutions of the Cauchy problem for a high-dimensional system of equations

$$\begin{cases} \dfrac{dx^n(t)}{dt} = A_n x^n(t), \quad t > 0, \\ x^n|_{t=0} = \widehat{x}^n, \end{cases} \quad A_{n+1} = \left.\begin{pmatrix} -n & 0 \dots & \dots & b \\ n & -n & \ddots & \ddots\ 0 \\ 0 & \ddots & \ddots & \ddots\ \vdots \\ \vdots & \ddots & \ddots & -n\ \ 0 \\ 0 & \dots & 0 & n\ \ a \end{pmatrix}\right\} n+1. \tag{2}$$

Let (a, b) belong to the domain of asymptotic stability of the zero solution of the equation from (1) (see [2]). Consider the Lyapunov equations

$$H_n A_n + A_n^* H_n = -F_n, \tag{3}$$

where F_n are positive definite symmetric matrices.

We establish that there exists a sequence of matrices F_n such that the sequence $\{H_n\}$ of solutions to (3) converges to a solution to a boundary value problem for a partial differential equation. We show that the sequence of the Lyapunov functions for (2) converge to a Lyapunov-Krasovskii functional for (1). Using this functional, we obtain estimates for the convergence rate of solutions to (1) to zero as $t \to \infty$.

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On Mixed Boundary Value Problems in the Quarter-Plane for the Vlasov System*

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We consider mixed boundary value problems in the quarter-plane for the Vlasov system. The boundary operators satisfy the Lopatinskii condition. The unique solvability of the mixed problems is established in anisotropic weighted Sobolev spaces.

Keywords: Vlasov system, pseudohyperbolic systems, mixed boundary value problems, anisotropic weighted Sobolev spaces, well-posedness

1 The main results

The report considers a class of mixed boundary value problems in the quarter-plane for the Vlasov system [1]:

$$\begin{cases} \begin{pmatrix} 1-\alpha_1 D_x^2 & \varepsilon \\ \varepsilon & 1-\alpha_2 D_x^2 \end{pmatrix} D_t^2 u + C D_x^4 u = f(t,x), \quad t>0,\ x>0, \\ u|_{t=0} = 0, \quad D_t u|_{t=0} = 0, \\ a_{j1} u_1 + a_{j2} D_x u_1|_{x=0} = 0, \quad j=1,2, \\ b_{j1} u_2 + b_{j2} D_x u_2|_{x=0} = 0, \quad j=1,2. \end{cases} \tag{1}$$

Here $u=(u_1,u_2)^T$, $f=(f_1,f_2)^T$, $\alpha_1,\alpha_2>0$, $0\le\varepsilon<1$, $C=\mathrm{diag}(c_1,c_2)$ with $c_1,c_2>0$. The constants a_{jk}, b_{jk} are real. This system is not resolved with respect to the highest time derivative and belongs to the class of pseudohyperbolic systems [2]. For problem (1), the Lopatinskii condition is assumed to be satisfied: $\det(a_{ij})\neq 0,\ \det(b_{ij})\neq 0$.

The main result is the proof of the well-posedness of this problem in anisotropic weighted Sobolev spaces $W^{2,4}_{2,\gamma}(\mathbb{R}^2_{++})$.

Theorem 1. *There exists $\gamma_0>0$ such that for any $f(t,x)\in W^{1,0}_{2,\gamma}(\mathbb{R}^2_{++})$, $\gamma>\gamma_0$ satisfying $f|_{t=0}=0$, problem (1) has a unique solution $u(t,x)\in W^{2,4}_{2,\gamma}(\mathbb{R}^2_{++})$ such that $D_t^2 D_x^2 u \in L_{2,\gamma}(\mathbb{R}^2_{++})$. The following estimate holds:*

$$\|u(t,x), W^{2,4}_{2,\gamma}(\mathbb{R}^2_{++})\| \le c(\gamma_0,\gamma)\|f(t,x), W^{1,0}_{2,\gamma}(\mathbb{R}^2_{++})\|.$$

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Estimation for the Mean Curve of Functional Data by a 3D-reproducing Kernel Hilbert Space Method

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The inference for mean curves with functional data has been widely used in biomedical, health science, meteorology and other fields. We propose new estimator for the mean curve of functional data by 3D-null space reproducing kernel hilbert space method. We then show the asymptotic properties of the proposed estimator. Our results can be effectively applied to inference for mean curve with more than two inflection points.

Keywords: functional data, 3D-reproducing kernel hilbert space, mean curve, reproducing

Let $\{Y(t),t \in (0,T]\}$ $(0 < T < \infty)$ be the stochastic process with the mean $\mu_0(t) = EY(t)$ and variance $\sigma^2(t) = E[\varepsilon^2(t)]$. It is obtained that i.i.d copies from it, $\{Y_i(t),t \in (0,T]\}$, $i = 1, \cdots, n$, then

$$Y_i(t) = \mu_0(t) + \varepsilon_i(t), t \in (0,T],$$

where $\varepsilon_i(\cdot)$ is i.i.d. measurement error with mean and variance, respectively as

$$E[\varepsilon_i(t)] = 0, E[{\varepsilon_i}^2(t)] = \sigma^2(t), (i = 1, \ \cdots, \ n)$$

The estimation of the mean function $\mu_0(\cdot)$ is considered using the reproducing kernel hilbert space method.

Let H be a hilbert space consisting of square integrable functions $\ell_2(T)$ on $[0, \ T]$, with a given inner product $< \cdot \ , \ \cdot >_H$. According to the definitoin of the reproducing kernel, $K(s, \ t) :=< K(s, \ \cdot), \ K(t, \ \cdot) >_H$ holds. K is not unique in given reproducing kernel hilbert space. A reproducing kernel under one inner product may not be a reproducing kernel under another inner product on the same space H. The adequate choice of kernel for specific statistical inference is important.

Let $\mu_0 \in H$ and $Kh = (Kh)(\cdot) =< K(\cdot\ ,), \ \ h >_H$, $h \in H$, then $\mu_0(\cdot)$ is estimated as follows ([3]):

$$\hat{\mu}_{n,\lambda}(\cdot) = \arg \inf_{\mu \in H} \{ \frac{1}{n} \sum_{i=1}^{n} \|Y_i - \mu\|^2 + \lambda J(\mu) \},$$

where, λ is a smoothing parameter and $J(\mu) = \|K\mu\|_H^2$ is a penalty functional for the kernel K. The null space and orthogonal space are denoted by $H_0 = \{h : h^{(3)}(\cdot) \equiv 0\}$, and $H_1 = \{h : h^{(j)}(0) = 0,\ j = 0, 1, 2\}$, respectively. Let $g_1, \cdots, g_d$ be orthogonal basis of H_0 and $d = \dim(H_0)$.

Theorem 1. Suppose that the inner product in this space denoted as follows:

$$< f,\ g >_{H,0} = \sum_{j=0}^{2} f^{(j)}(0) g^{(j)}(0),$$

$$< f,\ g >_{H,1} = \int_0^T f^{(2)}(t) g^{(2)}(t) dt,$$

then the reproducing kernel function is expressed as

$$K_0(s,\ t) = 1 + st + \frac{s^2 t^2}{4}, K_1(s,\ t) = \frac{(s \wedge t)^5}{120} - \frac{(s \vee t)(s \wedge t)^4}{24} + \frac{(s \vee t)^2 (s \wedge t)^3}{12} \tag{1}$$

Lemma 1. For $h \in H$, it holds that

$$(1)(K_0 h)(t) \ := < K_0(t\ ,\ s),\ h(s) >_H \in H_0 \tag{2}$$

$$(2)(K_1 h)(t) \ := < K_1(t\ ,\ s),\ h(s) >_H \in H_1 \tag{3}$$

Theorem 2. Assume $\mu_0(\cdot),\ Y_i(\cdot) \in H\ (i = 1, \ \ldots\ , n)$, then, for the given penalty functional $J(\mu)$ and fixed λ , there are constants $a = (a_1, \cdots, a_d)'$ and $b = (b_1, \cdots, b_n)'$ such that $\hat{\mu}_{n,\lambda}$ has the following representation

$$\hat{\mu}_{n,\lambda}(t) = \sum_{j=1}^{d} a_j g_j(t) + \sum_{i=1}^{n} b_i (K_1 Y_i)(t),\ \ t \in (0,\ T] \tag{4}$$

Theorem 3. Let two reproducing kernel functions be given by (6). Then $\hat{\mu}_{n,\lambda}(t)$ is expressed as

$$\hat{\mu}_{n,\lambda}(t) = \bar{Y}_n(t) - \frac{\lambda}{1+\lambda}(K_1 \bar{Y}_n)(t) = \bar{Y}_n(t) - \frac{\lambda}{1+\lambda}(-\frac{t^2}{2}\bar{Y}_n{}^{(2)}(0) - t\bar{Y}_n{}^{(1)}(0) + \bar{Y}_n(t) - \bar{Y}_n(0)) \tag{5}$$

Theorem 4. The estimator $\hat{\mu}_{n,\lambda}(t)$ of the mean curve has asymptotic inconvenience.

Theorem 5. Let $\mu_0(\cdot)$ be twice differentiable with the second derivative $\ddot{\mu}_0(\cdot)$. Then we have

$$n^{1/2}(\hat{\mu}_{n,\lambda}(t) - \mu_0(t) - b_n(t)) \xrightarrow{D} N(0, \sigma^2(t)), \tag{6}$$

where $\sigma^2(t) = Var(Y(t))$ and b_n is expressed as

$$b_n = \ddot{\mu}_0(t_j)(t_{j+1} - t)(t - t_j) - \frac{\lambda}{1+\lambda}(K_1 \bar{Y}_n)(t) + o(t_{j+1} - t_j)^2 \mu_0(\cdot),\ t \in [t_j, t_{j+1})$$

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Pattern Detection Method Using Phase Varying Dynamic Time Warping in Financial Time Series

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Dynamic time-warping is a matching method with computing similarity between two time series based on the elastic distance. DTW is typical method to measure similarity between time series subjected to nonlinear time warping, which is widely used in speech recognition, music information retrieval, motion analysis and gesture recognition. Based on the dynamic programming principle, the time complexity of DTW is O(N^2). We propose a novel phase-varying dynamic time-warping method (PDTW), which makes time series over-stretched and compressed in phase axis as well as time one. This method is applied to extract special patterns of finance time-series data.

Keywords: dynamic time warping, time series prediction, finance time series, pattern recognition

1 The main results

Let consider two continuous functions $y_p = f_p(t_p)$ and $y = f(t)$.

Where $y = f(t)$,$t \in [a_1, b_1]$, $\quad y \in [c_1, d_1]$ is a function for relatively long time measurements.

$y_p = f_p(t_p)$,$t_p \in [a_0, b_0]$, $\quad y \in [c_0, d_0]$ is a basic pattern function for a relatively short time.

For the graph of the basic pattern function $f_p(t_p)$, suppose the function specifying the scale of the transverse and longitudinal (phase)axes.

$$\varphi_t : [a_0, b_0] \to [a_1, b_1], \varphi_y : [a_1, b_1] \to R \tag{1}$$

Let $t = \varphi_t(t_p)$ is a monotonically increasing continuous function and

$$\varphi_t(a_0) = a_1, \quad \varphi_t(b_0) = b_1.$$

Similarly, $y = \varphi_y(t)$ is the phase gain function as a monotone function.

We should identify the existence of the shape of the underlying pattern in the graph of the measured data function and extract the position. We define the similarity measure of the measured data function $y = f(t)$ and the underlying pattern $f_p(t_p)$ as follows:

$$
\begin{aligned}
D(f_p, f) &= \min_{\varphi_y, \varphi_t \in \Phi} d(f_p, f : \varphi_y, \varphi_t) \\
d(f_p, f : \varphi_y, \varphi_t) &:= \left\| f_p - \varphi_y \cdot (f \circ \varphi_t) \right\|_{L^2(a_0, b_0)}
\end{aligned}
\tag{2}
$$

In case that data is given as two time series to be compared, consider numerical solution of (2).

If the data is discrete , optimization for $\varphi_t(t_p)$ and $\varphi_y(t)$ can be performed by generalizing the dynamic programming method used in the standard DTW method.

Extending the above method, we construct the PDTW method as follows.

We divide the gain region [a, b] of $\varphi_y(t)$ into K equal regions.

$$u_1 = a < u_2 <, ..., < u_K = b.$$

$$PDTW = (d_{ijk})_{M*N*K}.$$

Where d_{ijk} is following.

$$d_{i,1,k} = \sum_{l=1}^{i} |u_k f_p(t_l^p) - f(t_1)|, i \in [1 : M],\ k \in [1 : K]$$

$$d_{1,j,k} = |u_k f_p(t_1^p) - f(t_j)|, j \in [1 : N], k \in [1 : K]$$

$$
\begin{gathered}
d_{i,j,k} = \min \left\{ d_{i-1,j-1,k}, d_{i,j-1,k}, d_{i-1,j,k}, d_{i-1,j-1,k-1}, d_{i,j-1,k-1}, d_{i-1,j,k-1}, d_{i,j,k-1} \right\} + \\
+ |u_k f_p(t_i^p) - f(t_j)|, \\
i \in [2 : N], j \in [2 : M], k \in [2 : K].
\end{gathered}
$$

The position of the searched pattern is obtained to find local minima below a certain threshold among elements $i = M$ in the 3D matrix PDTW.

$$B = \arg\min_{j,k}(d_{M,j,k} < \tau).$$

For each element in $B = \{(M, j^{(1)}, k^{(2)}), \cdots, (M, j^{(L)}, k^{(L)})\}$, search the minimum pathto determine the position of the warping pattern.

Let assume that the search path is $Path = \{p(1), \cdots, p(V)\}$.

Where $p(v) = (i(v), j(v), k(v))$ is the vth 3D node of the path.

The path must satisfy the following properties:

(1) Monotonicity: Two adjacent nodes satisfy the following inequality:

$$i(v-1) \le i(v), \quad j(v-1) \le j(v), \quad k(v-1) \le k(v).$$

(2) Continuity: The value of i, j, k of the nodes of the path cannot jump more than 2.

i.e. $i(v) - i(v-1) \le 1, \quad j(v) - j(v-1) \le 1, \quad j(v) - j(v-1) \le 1.$

(3) Boundary conditions: The first and last points of the path satisfy the following conditions:

$$i(1) = 1, \quad i(V) = M.$$

Conclusion

We extend a subsequence dynamic time warping method that allows only changes in the longitudinal time axis to the phase-varying dynamic time warping which includes changes in the phase axis.

In addition, the proposed method is applied to a sine function with varying amplitude to verify the effectiveness.

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Determining the Unknown Source Term in Fractional Equations with Sequential Caputo Derivatives

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This study investigates the inverse problem of reconstructing a space-dependent source term in a fractional differential equation with sequential Caputo derivatives, utilizing an interior observation at a fixed time.

Keywords: inverse problem, Caputo fractional derivative, sequential fractional derivative, bivariate Mittag-Leffler function

Fractional equations are useful for modeling systems with long-term memory, but simple ones struggle with complex, layered processes. To solve this, models use sequential fractional derivatives. This simply means applying the math operations one after another, where the result of one step becomes the starting point for the next [1,2].

We consider the sequential fractional differential equation on $\Omega = (0,1) \times (0,T]$:

$$D_{0t}^{\beta}(D_{0t}^{\alpha} u(x,t)) + D_{0t}^{\beta} u(x,t) = u_{xx}(x,t) + g(x)f(t),$$

subject to given initial states for $u(x,0)$ and $D_{0t}^{\alpha} u(x,0)$, alongside zero Dirichlet boundary conditions at $x = 0$ and $x = 1$. Here, D_{0t}^{γ} represents the Caputo time-fractional derivative of order $0 < \gamma < 1$. To determine both the state $u(x,t)$ and the unknown spatial source $g(x)$, we utilize an additional observation at a fixed time $t_0 \in (0,T)$:

$$u(x,t_0) = \rho(x), \quad 0 \leq x \leq 1,$$

assuming all given functions are known and sufficiently smooth.

Exact analytical solutions are constructed using a Fourier sine series spectral expansion. Furthermore, by leveraging the properties and asymptotic behaviour of the bivariate Mittag-Leffler function [3,4], the existence, uniqueness, and uniform convergence of the classical solutions are rigorously established within the framework of Hölder spaces [5,6].

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A Non-monotone Conic Trust Region Method with Non-monotone Line Search for Solving Unconstrained Optimization Problems

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The paper presents the new non-monotone term that can be regarded as an extension of former non-monotone term. Updating parameters in non-monotone term is done to control the degree of non-monotone reasonably. The new method retains the global convergence under some reasonable assumptions.

Keywords: non-monotone technique, line search, conic model, conic trust region method, unconstrained optimization problems

1 The main results

I propose a new Non-monotone term defined as follows:

$$P_k = \max\left\{\bar{P}_k,\ f_k\right\}$$

$$\begin{aligned}\bar{P}_k = {} & (1-\eta_k)\, f_k + \eta_k\,(1-\eta_{k-1})\, f_{k-1} + \eta_k\eta_{k-1}\,(1-\eta_{k-2})\, f_{k-2} + \cdots \\ & \cdots + \eta_k\eta_{k-1}\cdots\eta_{k-m_k+2}\left(1-\eta_{k+m_k+2}\right) f_{k-m_k+1} + \eta_k\eta_{k-1}\cdots\eta_{k-m_k+1} f_{k-m_k}\end{aligned} \tag{1}$$

$$m_k = \min\left\{m_{k-1}+1,\ \delta_M + M,\ M_k\right\} \tag{2}$$

where $m_0 = 0$, $M_0 = M$, $0 < \eta_i < 1$, $(i = 1, 2, \cdots)$. M_k and M are integers and δ_M is variation of M, which represents the interval of m_k. Obviously, the bigger m_k is, the stronger its non-monotonousness will be. Also,

$$(1-\eta_k) + \eta_k(1-\eta_{k-1}) + \cdots + \eta_k\eta_{k-1}\cdots\eta_{k-m_k+2}(1-\eta_{k-m_k+1}) + \eta_k\eta_{k-1}\cdots\eta_{k-m_k+1} = 1 \tag{3}$$

So $\bar{P}_k$ is a convex linear combination of f_k and m_k objective function values.
If $m_k = 0$, we set $P_k = f_k$. Then, the algorithm goes to the monotone step.
Similar to the case of quadratic model sub-problem, we use the following ratio

$$r_k = \frac{rared(s_k)}{pred(s_k)} = \frac{P_k - f(x_k + s_k)}{\phi_k(0) - \phi_k(s_k)} \tag{4}$$

If the trial step is rejected, line search is done. It is assumed that the step from line search should satisfy the following

$$g_k^T s_k \leq -a_1 \|g_k\|^2 \tag{5}$$

$$\|s_k\| \leq a_2 \|g_k\| \tag{6}$$

,where a_1 and a_2 are positive.

However, Eq. (1) makes it difficult to calculate $\bar{P}_k$ because $m_k(1 + m_k)/2$ multiplications are needed, so we are going to give recursions.

In the algorithm, we calculate M_k by comparing the ratio r_k with a constant $0 < \alpha_1 < \alpha_2 < 1$ as follows:

$$M_{k+1} = \begin{cases} M_k + 1, & r_k \leq \alpha_1 \\ M_k, & \alpha_1 < r_k < \alpha_2 \\ M, & r_k \geq \alpha_2 \end{cases} \tag{7}$$

In this paper, we set $\eta_1 = 0.5$ and for $k \geq 1$, we update η_k by following formula:

$$\eta_{k+1} = \frac{2}{\pi} \arctan \frac{P_k - f_{k+1}}{P_{k-1} - f_k} \tag{8}$$

In the previous trust region algorithm, the computation stops if the gradient of function is zero, so the chance of finding the direction of search is missed if the current point is saddle. Taking this point into account, we propose a novel non-monotone cone trust region algorithm combining the non-monotone heuristics studied in the above subsection. So I added a new stopping condition that stops the computation, if the gradient of the function is zero and B_k is a positive matrix, then x_k is an approximate solution.

This algorithm can control non-monotonousness and based on a combination of coinic model and line search, so it have several advantages.

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On the Asymptotic Stability of Solutions to a Class of Nonlinear Delay Difference Equations with Periodic Coefficients in Linear Terms

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We consider a class of systems of nonlinear difference equations with periodic coefficients in linear terms and a variable delay. Conditions for the asymptotic stability of the zero solution are established and estimates characterizing stabilization rates of solutions at infinity are obtained. We use a special Lyapunov–Krasovskii functional.

Keywords: delay difference equations, asymptotic stability, Lyapunov–Krasovskii functional, solution estimates

We consider a system of nonlinear difference equations with a variable delay

$$x_{n+1} = A(n)x_n + B(n)x_{n-\tau(n)} + F(n, x_n, x_{n-1}, \dots, x_{n-\tau}), \qquad n = 0, 1, \dots, \tag{1}$$

where $A(n)$, $B(n)$ are $m \times m$ matrices, $A(n) = A(n+N)$, $B(n) = B(n+N)$, $N \in \mathbb{N}$, $\tau(n) \in \mathbb{N}$ is a variable delay, $1 \le \tau(n) \le \tau < \infty$, the continuous vector-function $F(n,u) = F(n, u_0, u_1, \dots, u_\tau)$ satisfies the inequality

$$\|F(n,u)\| \le q_0\|u_0\|^{1+\omega_0} + \dots + q_\tau\|u_\tau\|^{1+\omega_\tau}, \quad q_i,\ \omega_i \ge 0. \tag{2}$$

Using a special Lyapunov–Krasovskii functional, the asymptotic stability of solutions to linear systems of the form (1) (with $A(n) \equiv A$, $B(n) \equiv B$, $F(n,u) \equiv 0$) was investigated in [1]. The functional has the following form

$$\langle Hx_n, x_n\rangle + \sum_{j=1}^{\tau}\langle K_{j-1}x_{n-j}, x_{n-j}\rangle, \quad H = H^* > 0, \quad K_j = K_j^* > 0, \quad j = 0, \dots, \tau, \tag{3}$$

and it is a discrete analogue of the functional proposed in [2] to study the asymptotic stability of the zero solution to systems of delay differential equations. The cases $\omega_0 > 0$, $q_i = 0$, $i = 1, \dots, \tau$, and $\omega_j = 0$, $j = 0, \dots, \tau$ (with $A(n) \equiv A$, $B(n) \equiv B$), were studied in [3] by using the functional (3).

Using a special Lyapunov–Krasovskii functional of the form

$$\langle H(n)x_n, x_n\rangle + \sum_{j=1}^{\tau}\langle K_{j-1}x_{n-j}, x_{n-j}\rangle,$$
$$H(n) = H^*(n) > 0, \quad H(n) = H(n+N), \quad K_j = K_j^* > 0, \quad j = 0, \dots, \tau, \tag{4}$$

the asymptotic stability of solutions to systems of the form (1) (with $F(n,u) \equiv 0$) was investigated in [4]. Using the same functional the cases $\omega_j = 0$, $j = 0, \dots, \tau$, and $\omega_0 > 0$,

$q_i = 0$, $i = 1, \ldots, \tau$, were studied in [5,6] respectively. The goal of this work is to study the asymptotic stability of solutions to (1) when $F(n, u)$ satisfies (2).

By using a special Lyapunov–Krasovskii functional, we obtain conditions for the asymptotic stability of the zero solution to nonlinear systems of the form (1). Estimates for attraction sets and decay rates of solutions to (1) at infinity are obtained.

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Analytical and Numerical Solutions with Zero Fronts to a Degenerate Parabolic Predator-Prey System*

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The paper deals with a quasilinear second-order parabolic system, which have been proposed by Murray as a model of two species population dynamics. Solutions with zero fronts are the objects under study. A theorem of existence and uniqueness of analytical solution under a boundary condition specified in a certain point is constructively proved. The solution is constructed in the form of multiple power

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series with recurrently determined coefficients. The zero front equation in the form of power series is constructed simultaneously. A new class of exact solutions is found by the reduction of the original system to a system of ordinary differential equations. A numerical solution algorithm is proposed based on the collocation method and radial basis function approximation. Numerical analysis is fulfilled to estimate the series convergence domains and to verify the numerical algorithm.

Keywords: degenerate quasilinear parabolic system, existence and uniqueness theorem, power series, exact solution, numerical solution, collocation method, radial basis function

We consider a degenerate parabolic system which have been proposed by J.D. Murray [1] as a predator-prey model:

$$u_t = \alpha_1 u_x + \beta_1(uv_{xx} + v_x u_x) + F(u,v),\ v_t = \alpha_2 v_x - \beta_2(vu_{xx} + u_x v_x) + G(v,u). \quad (1)$$

Here, u and v are the desired functions, F and G are known sufficiently smooth functions.

From the population biology point of view it is advisable to consider solutions with two zero fronts (see Fig. 1) which satisfy the following conditions:

$$u(t,x)|_{x=a(t)},\ v(t,x)|_{x=b(t)} = 0, \quad (2)$$

where $x = a(t)$, $x = b(t)$ are the equations of zero fronts. The functions $a(t)$, $b(t)$ can be either given or unknown [2].

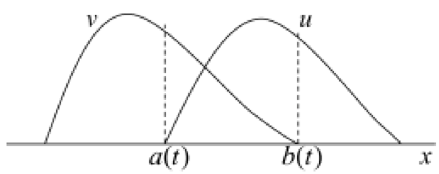

Figure 1: The interaction of preys (v) and predators (u)

On each zero front the parabolic type of system (1) is degenerate. If $F(0,0) = 0,\ G(0,0) = 0$, then system (1) has the solution $u \equiv 0$, $v \equiv 0$, and we can consider a non-negative solution with zero fronts as a part of a diffusion wave.

The present work is devoted to the construction of solutions to system (1) under the boundary conditions specified in point $x = 0$,

$$u(t,x)|_{x=0} = f(t),\ v(t,x)|_{x=0} = g(t), \quad (3)$$

which satisfies to (2) for some a priori unknown functions $a(t)$, $b(t)$.

Theorem 1. *Assume that*

1. $f(t),\ g(t)$ *are analytical functions in some neighborhood of point* $t = 0$;
2. $F(u,v),\ G(u,v)$ *are analytical functions in some neighborhood of point* $u = 0,\ v = 0$;
3. $f(0) = g(0) = 0$, $f'(0) = g'(0) = 0$, $f''(0) > 0,\ g''(0) > 0$, $F(0,0) = G(0,0) = 0$.

Then, the problem (1), (3) *in some neighborhood of point* $t = 0,\ x = 0$ *has two analytical solutions, one of which have the form of diffusion wave.*

In the proof of the theorem, we construct an analytical solution in the form of multiple power series

$$\begin{aligned} u(t,x) &= \sum_{k,l=0}^{\infty} u_{k,l}\frac{t^k x^l}{k!l!},\ u_{k,l} = \frac{\partial^{k+l}u}{\partial t^k x^l}\Big|_{t=0,x=0}; \\ v(t,x) &= \sum_{k,l=0}^{\infty} v_{k,l}\frac{t^k x^l}{k!l!},\ v_{k,l} = \frac{\partial^{k+l}v}{\partial t^k x^l}\Big|_{t=0,x=0}. \end{aligned} \tag{4}$$

The coefficients of the series (4) are determined by recurrence formulas, the convergence is proved by the classical majorant method [3].

In the particular case, when $\alpha_1 = \alpha_2 = \alpha$, and $F(u,v) = Av^\lambda u^{\gamma-\lambda},\ G(v,u) = Bu^\mu v^{\gamma-\mu}$, where $A,\ B,\ \lambda,\ \mu,\ \gamma$ are constants, such that $\gamma > \lambda > 0$, $\gamma > \mu > 0$, system (1) is reduced to a system of ordinary differential equations (SODE) with the use of non-classical variables separation. The solution of the SODE leads to the construction of exact solutions of system (1) with zero fronts.

To construct a numerical solution to problem (1), (3), we use finite-difference time discretization. At each time step we solve iteratively the system of corresponding elliptic equations using the collocation method and radial basis functions.

The calculational experiment and numerical analysis of the constructed solutions allows us to estimate in particular cases the domain of series convergence. The comparison of the numerical solutions with the exact ones allows to verify the numerical algorithm and to estimate its correctness.

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Fixed Point Methods in Bilinear Optimal Control Problems*

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In the class of bilinear optimal control problems, conditions for optimality and nonlocal control improvement are developed in the form of special fixed-point problems for control operators. Control improvement methods are constructed for finding extremal controls based on solving the constructed fixed-point problems.

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The proposed fixed-point methods eliminate the labor-intensive operation of varying the control at each iteration, unlike gradient methods and other well-known methods for local control improvement. Convergence of the constructed iterative processes with respect to the residual of the maximum principle is proven. The comparative effectiveness of fixed-point methods is demonstrated using model problems.

Keywords: bilinear optimal control problem, optimality conditions and control improvements, fixed point problem, iterative methods

1 Bilinear optimal control problem and iterative methods

In [1], based on the construction of non-standard formulas for incrementing the objective functional that do not contain residual terms of the expansions, effective methods for non-local control improvement in bilinear control systems with quadratic control quality functionals are developed. Control improvement is achieved by solving special Cauchy problems for phase and conjugate systems.

In [2], fixed point methods were developed in a class of nonlinear optimal control problems. In this paper, new methods for finding extremal controls in the considered class of bilinear optimal control problems are constructed based on the representation of control improvement conditions in the form of fixed point problems in control space.

The class of bilinear optimal control problems is considered:

$$\Phi(u) = \varphi(x(t_1)) + \int_T F(x(t), u(t), t)dt \to \inf_{u \in V}, \tag{1}$$

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x^0, \quad u(t) \in U \subset R^m, \quad t \in T = [t_0, t_1], \tag{2}$$

in which the function $\varphi(x)$ is linear on R^n, the functions $f(x, u, t)$, $F(x, u, t)$ are linear in the variable x, linear in the variable u and continuous in the variable t on the set $R^n \times U \times T$. As admissible controls $u(t) = (u_1(t), \ldots, u_m(t))$ the set V of piecewise continuous functions on the interval T with values in the compact and convex set $U \subset R^m$ is considered. The initial state x^0 and the interval T are fixed.

In this paper, we use general notations of bilinear functions $f(x, u, t)$, $F(x, u, t)$ for simplicity and convenience of presentation of the constructions of the proposed methods, as well as the general designation of the linear function $\varphi(x)$ for uniformity with the designation of the specified functions.

The Pontryagin function with the adjoint variable ψ and the standard adjoint system in general notation have the form:

$$H(\psi, x, u, t) = \langle \psi, f(x, u, t) \rangle - F(x, u, t), \quad \psi \in R^n,$$

$$\dot{\psi}(t) = -H_x(\psi(t), x(t), u(t), t), \quad t \in T, \quad \psi(t_1) = -\varphi_x(x(t_1)). \tag{3}$$

For an admissible control $v \in V$ denote $x(t, v)$, $t \in T$ as the solution to system (1); $\psi(t, v)$, $t \in T$ as the solution to the standard adjoint system (3) for $x(t) = x(t, v)$, $u(t) = v(t)$.

Let us define the mapping u^α with parameter $\alpha > 0$ using the relation:

$$u^\alpha(\psi, x, w, t) = PU(w + \alpha H u(\psi, x, w, t)), \ x \in R^n, \ \psi \in R^n, \ w \in U, \ t \in T.$$

Using the mapping u^α the well-known condition of the maximum principle [1] can be written as:

$$u(t) = u^\alpha(\psi(t, u), x(t, u), u(t), t), \quad t \in T. \tag{4}$$

Let us consider the problem of improving the control $u \in V$: find the control $v \in V$ with the condition $\Phi(v) \leq \Phi(u)$. The paper proposes two conditions for improving control $u \in V$ which are considered as fixed point problems:

$$v(t) = u^{\alpha}(\psi(t,u), x(t,v), u(t), t), \quad t \in T. \tag{5}$$

$$v(t) = u^{\alpha}(\psi(t,v), x(t,u), u(t), t), \quad t \in T. \tag{6}$$

To solve the fixed point problems (5) and (6), iterative processes are constructed for $k \geq 0$ with a given initial approximation $v^0 \in V$ for $k = 0$:

$$v^{k+1}(t) = u^{\alpha}(\psi(t,u), x(t,v^k), u(t), t), \quad t \in T. \tag{7}$$

$$v^{k+1}(t) = u^{\alpha}(\psi(t,v^k), x(t,u), u(t), t), \quad t \in T. \tag{8}$$

The convergence of the iterative processes (7) and (8) to the solution of problem (4) under certain conditions is proved.

Let us highlight the main features of the proposed fixed point methods.

1. Non-locality of control improvement and the absence of a labor-intensive operation of convex or needle-shaped control variation at each iteration, which is typical for gradient methods.

2. The possibility of strictly improving extremal controls in contrast to gradient and other known projection methods of local control improvement.

The indicated properties of the methods are important factors for increasing the efficiency of optimization of bilinear control systems.

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On Constructing a Particular Solution of the Riccati Equation with Arbitrary Coefficients

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The paper investigates the problems of constructing particular solutions of the Riccati equation. It is established that there exists a class of coefficient-conjugate Riccati equations for which a particular solution can be found by solving a linear nonhomogeneous differential equation or a Bernoulli differential equation. A procedure for transforming the general Riccati equation into a coefficient-conjugate Riccati equation is presented. The theoretical results are confirmed by examples.

Keywords: Riccati equation, integral identity, coefficient-conjugate equations, fractional-linear transformation, Jacobi determinant, algorithms for constructing a particular solution

1 Main Results

Since its appearance (1724), the Riccati equation has attracted the attention of mathematicians because, despite the abundance of studies (see bibliography [1]), no method for constructing its particular solution has been developed to date. Since the second half of the 20th century, with the emergence of optimal control theory, interest in the Riccati equation has increased even more, since the solvability of many optimization problems, especially in solving the problem of optimal control synthesis, reduces to the solvability of the Riccati equation [1].

The paper studies the problem of constructing particular solutions of the Riccati equation with arbitrary coefficients

$$z'(x) = k_0(x) + k_1(x)z(x) + k_2(x)z^2(x), \tag{1}$$

where the coefficients $k_0(x)$, $k_1(x)$, and $k_2(x)$ are continuous functions. In the course of the study, it is established that there exists a class of coefficient-conjugate Riccati equations for which a particular solution can be found by solving a linear nonhomogeneous differential equation

$$y_0'(x) - k_1(x)y_0(x) = k_0(x), \tag{2}$$

or a Bernoulli differential equation

$$y_0'(x) - k_1(x)y_0(x) = k_2(x)y_0^2(x). \tag{3}$$

A procedure for transforming the general Riccati equation into a coefficient-conjugate Riccati equation is presented. Equations of type (1) are called equations of the class of coefficient-conjugate Riccati equations if the coefficients $k_0(x)$, $k_1(x)$, and $k_2(x)$ satisfy the integral identity

$$\int k_0(x)e^{-\int k_1(x)\,dx}\,dx \int k_2(x)e^{\int k_1(x)\,dx}\,dx = 2. \tag{4}$$

An algorithm for constructing a particular solution of the Riccati equation with arbitrary continuous coefficients is developed.

On the other hand, the Riccati equation is closely related to linear differential equations of the second order and linear systems of differential equations with variable coefficients, for which methods for constructing their particular solutions have also not been developed [1]. In this regard, the paper also presents in detail a procedure for determining particular solutions of linear differential and integro-differential equations [2] with variable coefficients, which are in one way or another related to the Riccati equation.

Thus, the developed method for determining a particular solution of the Riccati equation makes it possible to analytically find the general solution and to carry out qualitative studies of a fairly wide class of linear differential and integro-differential equations and their systems with variable coefficients.

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Subgradient and Finite-Difference Stochastic Methods for Quasiconvex Optimization Problems: Convergence Rate

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Keywords: quasi convex function, convergence rate, normalized subgradient

1 Main Results

Quasiconvex optimization problems arise in a wide range of applications and have been extensively studied as a natural generalization of convex optimization. Early convergence results for minimizing sequences in quasiconvex problems were obtained by Polyak [1], who introduced the notion of strong quasiconvexity and established fundamental existence and convergence theorems. Further properties of strongly quasiconvex functions were investigated in [3], while algorithmic aspects and proximal-type methods were studied more recently in [2].

Nonsmooth quasiconvex optimization methods have also attracted considerable attention. In particular, Nesterov [4] proposed minimization schemes for nonsmooth and quasiconvex functions, extending classical subgradient ideas beyond the convex framework. However, many convergence results in the literature rely on convex subdifferentials or smoothness assumptions that may be restrictive for practical applications.

In this paper, we develop a subgradient projection method for quasiconvex minimization based on the Clarke subdifferential [5], which is well suited for locally Lipschitz and nonsmooth functions. Under assumptions that are standard in convex optimization, we establish convergence of the generated iterates for strongly quasiconvex objective functions in the sense of Polyak [1]. Moreover, we derive explicit convergence rates for both the iterates and the objective function values, thereby extending several known results from convex optimization to the quasiconvex setting.

Finally, we investigate a finite-difference stochastic scheme for constrained quasiconvex optimization problems with inequality constraints. For the proposed zeroth-order method, we obtain a convergence rate of order $O(k^{-1/4})$, which is consistent with known complexity bounds for stochastic finite-difference methods.

The following notation is used throughout the paper:

(x, y) denotes the scalar product of vectors $x, y \in \mathbb{R}^n$.

$f'(x)$ denotes the gradient of f at the point x.

When ξ is a random vector, the notation $E[\xi]$ refers to its mathematical expectation.

1.1 Strongly quasiconvex functions and Clarke subdifferential

Definition 1 (Strong quasiconvexity [1]). *A function f is said to be* strongly quasiconvex *on a convex set $M \subset \mathbb{R}^n$ if there exists a constant $\mu > 0$ such that for all $x, y \in M$ and all $\lambda \in [0,1]$,*

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} - \mu\lambda(1-\lambda)\|x-y\|^2.$$

Definition 2 (Clarke subdifferential, [5]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke subdifferential of f at a point x is defined as*

$$\partial f(x) = \{v \in \mathbb{R}^n : f^\circ(x; h) \geq \langle v, h\rangle \text{ for all } h \in \mathbb{R}^n\},$$

where $f^\circ(x; h)$ denotes the generalized directional derivative

$$f^\circ(x; h) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda h) - f(y)}{\lambda}.$$

1.2 Subgradient projection method

We consider the following subgradient projection algorithm for minimizing a strongly quasiconvex function over a closed convex set M:

$$x^{k+1} = \Pi_M(x^k - \rho_k v^k), \qquad v^k \in \partial f(x^k),$$

where Π_M denotes the Euclidean projection onto M and $\{\rho_k\}$ is a sequence of positive step sizes.

1.3 Convergence of iterates

We first establish convergence rates for the sequence of iterates generated by the algorithm.

Theorem 1. *Suppose that:*

(i) The function f is locally Lipschitz and μ-strongly quasiconvex on the set M.
(ii) The step sizes are chosen as $\rho_k = 1/k$.

Then the sequence $\{x^k\}$ generated by the subgradient projection method satisfies the following estimates:

$$\|x^k - x^*\|^2 = \begin{cases} O\left(\frac{1}{k}\right), & \mu > 1, \\ O\left(\frac{\ln k}{k}\right), & \mu = 1, \\ O\left(\frac{1}{k^\mu}\right), & 0 < \mu < 1, \end{cases}$$

where x^* is a solution of the minimization problem.

1.4 Convergence of objective values

Next, we derive convergence rates for the objective function values.

Theorem 2. *Under the assumptions of Theorem 1, the sequence of objective values satisfies*

$$\min_{1 \leq k \leq T}(f(x^k) - f(x^*)) = \begin{cases} O\left(\frac{\ln T}{T}\right), & \mu > 1, \\ O\left(\frac{\ln^2 T}{T}\right), & \mu = 1, \\ O\left(\frac{1}{T^\mu}\right), & 0 < \mu < 1. \end{cases}$$

1.5 Finite-difference stochastic scheme for constrained problems

We now consider constrained quasiconvex optimization problems of the form

$$f_0(x) \to \min, \qquad x \in M, \quad f_i(x) \le 0, \quad i = 1, \dots, m.$$

Define the aggregated constraint function

$$\tilde{f}(x) = \max_{1 \le i \le m} f_i(x).$$

Since gradient information is unavailable, we employ a finite-difference stochastic approximation of the gradient given by

$$g(x, \alpha, u) = \frac{n}{2\alpha}[f(x + \alpha u) - f(x - \alpha u)]u, \tag{1}$$

where u is either uniformly distributed on the unit sphere in $\mathbb{R}^n$ or follows a standard Gaussian distribution.

The following theorem establishes a convergence rate for the proposed finite-difference stochastic scheme.

Theorem 3. *Assume that:*

(i) Each function f_i is Lipschitz continuous with constant L, μ-strongly quasiconvex on M, and has Lipschitz continuous gradients with constant L'. Moreover, assume $f_i'(x) \neq 0$ for all $x \in M$ and define

$$\gamma = \min_{1 \le i \le m} \min_{x \in M} \|f_i'(x)\| > 0.$$

(ii) The sequences $\rho_k = 1/\sqrt{k + 0.5}$ and $\alpha_k = 1/(k+1)$ are used, and let $\tilde{R}^2 = \max\{R^2,\, n^2 L^2\}$.

(iii) The iterates are generated by

$$x^{k+1} = \Pi_M(x^k - \rho_k p^k),$$

where p^k is chosen according to the rule described above.

Then the expected objective residual satisfies

$$\mathbb{E}\Big[\min_{1 \le i \le k} f(x^i) - f(x^*)\Big] = O\big(k^{-1/4}\big)\,,$$

where x^* is any solution of the constrained problem.

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On Markov Property for Any Solution of Nonlocal Continuity Equation

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A distributional solution of controlled nonlocal continuity equation can be realized by a suited Markov process.

Keywords: continuity equation, Markov property, the deterministic mean field type control problem

Consider a phase space $\mathbb{R}^d$ and open bounded time interval I. Let a control set U be a closed subset of a separable Banach space. Denote its Borel σ-algebras by $\mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$, and $\mathcal{B}(U)$. Fix a given $p > 1$ and denote by $\mathcal{P}^p(\mathbb{R}^d)$ the set of probabilities over $\mathbb{R}^d$ with the finite p-th moment.

Following [1], for a given Borel velocity field $v(t,x)$ on $I \times \mathbb{R}^d$ we say that an arc of probability measures $m(t)$ over $\mathbb{R}^d$ $(t \in I)$ is a distributional solution of the continuity equation

$$\partial_t m(t) + \operatorname{div}(v(t,x)m(t)) = 0 \qquad (*)$$

iff, for every $\varphi \in C_c^\infty(I \times \mathbb{R}^d)$, continuous real function with compact support, one has

$$\int_I \int_{\mathbb{R}^d} [\partial_t \varphi(t,x) + \nabla_x \varphi(t,x) v(t,x)] m(t,dx)\, dt = 0.$$

Consider a given Borel function $f : I \times \mathbb{R}^d \times \mathcal{P}^p(\mathbb{R}^d) \times U \to \mathbb{R}^d$ as dynamics function.

We say that the pair $(m(\cdot), u_E)$ is an Eulerian control process [3] if u_E is in $L^p(I \times \mathbb{R}^d, \mathcal{B}(I \times \mathbb{R}^d), \lambda; U)$ and $m(\cdot) \in (AC)^p(I; \mathcal{P}^p(\mathbb{R}^d))$ is a distributional solution of the nonlocal continuity equation $(*)$ with velocity field $v_E(t,x) = f(t,x,m(t),u_E(t,x))$.

Fix also a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the probability $\mathbb{P}$ has no atoms.

We say that the pair (X, u_L) is a Lagrangian control process [3] if $X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; C(I; \mathbb{R}^d)$, $u_L \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(I, \mathcal{B}(I), \lambda; U))$, and, for all $\omega \in \Omega$, $X(\cdot, \omega)$ is absolutely continuous and solves

$$\frac{d}{dt} X(t,\omega) = f(t, X(t,\omega), X(t)\sharp\mathbb{P}, u_L(t,\omega)) \text{ a.e.}$$

It was recently show that any Eulerian control process $(m(\cdot), u_E)$ can be realized by some Lagrangian control process (X, u_L); in particular, this provided $m(t) = X(t)\sharp\mathbb{P}$ for every $t \in I$. However, the corresponding proofs [2], [3] require the convexity of U, the Lipschitz continuity of f in (x, m), and the affinity of f in u. We propose, on the one hand, to discuss the possibility of significantly weakening these assumptions, and, on the other hand, to require that X become a Markov process.

Claim: under mild assumptions, any Eulerian control process $(m(\cdot), u_E)$ can be realized a suited Markov Lagrangian control process (Y, u_M) on $(\Omega, \mathcal{F}, \mathbb{P})$.

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Unbounded Viscosity Solutions of Hamilton-Jacobi-Bellman Equation for the Finite Horizon Optimal Control Problem

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In this paper we considered the finite horizon optimal control problem with growth conditions for control systems and cost functionals. We prove the uniqueness of the unbounded viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation and the convergence of the approximation scheme.

Keywords: value function, Hamilton-Jacobi-Bellman Equation, viscosity solution, uniqueness, approximation scheme, convergence

1 Introduction

Finding optimal feedback control law for an optimal control problem is a major issue in optimal control theory. Numerical solution is only the feasible way to get practically the optimal control law. Efforts in solving numerically the HJB equation for finite horizon optimal control problems can be found in [1,2]. The research focused on a general deterministic system with finite horizon cost functional without discount factor [3].

The aim of this paper is to develop the uniqueness of the viscosity solution and the convergence of the approximation scheme by relaxing the boundedness condition for the control system and the cost functional to the growth condition. For control systems, relaxing to the local Lipschitz continuity condition and growth condition does not guarantee the unique existence of a global solution of the control system corresponding to admissible control. Also, since the value function is unbounded, it is difficult to prove the comparison principle of unbounded viscosity solutions of the Hamilton-Jacobi-Bellman equation.

2 The main result

For a given $T > 0$, we consider the following finite horizon optimal control problem.

$$\begin{cases} \dot{y}(\tau) = f(\tau, y(\tau), u(\tau)), t < \tau \le T \\ y(t) = x \\ u(\tau) \in U \\ J(t, x, u(.)) = \int_t^T \ell(\tau, y(\tau), u(\tau))d\tau + g(y(T)) \to \min\limits_{u(\cdot)} \end{cases} \tag{1}$$

where $u(\cdot) \in \mathcal{U} = \{u(\cdot) : [0, T] \to U | u : measurable\}$.
In all the paper, we assume the following
(**A**1). $f : [0, T] \times R^n \times U \to R^n$ is continuous and satisfies the following condition:

$$\forall R > 0, \quad \exists L_f(R) > 0, \quad \forall t, s \in [0, T], \quad \forall x, y \in \bar{B}(0, R), \quad \forall u \in U,$$
$$\| f(t, x, u) - f(s, y, u) \| \le L_f(R)(|t - \varepsilon| + \| x - y \|).$$

(**A**2). $f : [0, T] \times R^n \times U \to R^n$ satisfies the following condition:

$$\exists C > 0, \quad \forall t \in [0, T], \quad \forall x \in R^n, \quad \forall u \in U, \quad (f(t, x, u), x) \le C(1 + \| x \|^2).$$

(**A**3). $f : [0, T] \times R^n \times U \to R^n$ satisfies the growth condition:

$$\forall t \in [0, T], \quad \forall x \in R^n, \quad \forall u \in U, \quad \| f(t, x, u) \| \le C(1 + \| x \|^p),$$

where $p \ge 1$ is constant and $\| \cdot \|$ represents Euclidian norm in R^n.
(**A**4). $\ell : [0, T] \times R^n \times U \to R$ is continuous and $g : R^n \to R$ is continuous and satisfies the following condition:

$$\exists C > 0, \quad \forall t \in [0, T], \quad \forall x \in R^n, \quad \forall u \in U, \quad |\ell(t, x, u)| \le C(1 + \| x \|^2),$$
$$\exists C > 0, \forall x \in R^n, |g(x)| \le C(1 + \| x \|^2).$$

When (A1), (A2) hold, the unique existence of a global solution of the control system corresponding to admissible control follows. In problem 1 value function is defined as

$$v(t, x) = \inf_{u \in \mathcal{U}} J(t, x, u)$$

and satisfies the dynamic programming principle.
Lemma 1. Assume that $v(t, x) \in E([0, T] \times R^n)$ is a viscosity solution of HJB equation. Then for any $\alpha > 0$, $\quad w(t, x) = v(t, x)e^{-\alpha\xi(x)}$ is a viscosity solution of

$$-\frac{\partial u(t, x)}{\partial t} + \bar{H}(t, x, D_x u(t, x)) = 0, \tag{2}$$

where $\bar{H}(t, x, D_x u(t, x)) = \sup\limits_{u \in U}\{-f(t, x, u) \cdot (\alpha u(t, x)D\xi(x) + D_x u(t, x)) - e^{-\alpha\xi(x)}\ell(t, x, u)\}$.
Theorem 2. Suppose (A1)-(A4). Then if upper semicontinuous function $v_1 \in E([0, T] \times R^n)$ is a viscosity sub-solution of HJB equation, lower semicontinuous function $v_2 \in E([0, T] \times R^n)$ is a viscosity super-solution of and

$$v_1(T, x) \le v_2(T, x), \quad x \in R^n,$$

we have

$$v_1(t,x) \le v_2(t,x), \quad (t,x) \in [0,T] \times R^n.$$

Lemma 3. Assume (A1)-(A4) for the optimal control problem 1. Then $\underline{v}(t,x), \bar{v}(t,x)$ are respectively viscosity sub- and super-solution of HJB Equation.

Theorem 4. Assume (A1)-(A4). Let $v(t,x)$ be the value function of optimal control problem1, then if $g(x)$ is continuous, the solution v_h^k of time-space discretized problem converges uniformly to v when $h \to 0, \quad k \to 0$ for any compact set $K \subset R^n$ in $[0,T] \times K$. If function g satisfies $(g^*)_* = g_*$, $(\bar{v})_* = \underline{v} = v$ holds and if $v(t,x)$ is continuous in K for any $t \in [0,T)$ and the compact subset $K \subset R^n$, when $h \to 0, k \to 0$, v_h^k converges to v in K.

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Importance Sampling Combining the Subset Simulation, Scaling of PDF and Sequential Importance Sampling

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By integrating the parameter in subset simulation, the scaling parameter of PDF, and the parameter in SIS, a one-parameter control is implemented. Also, by approximating the indicator function in the SIS method to a new function that has the same effect rather than a parameterized standard normal distribution function, the probability estimation time is reduced.

Keywords: subset simulation, scaling of PDF, sequential importance sampling, probability estimation, importance sampling, limit state function

1. Introduction and three importance sampling methods

The sequential importance sampling (SIS) using Gaussian mixture is discussed and its performance is compared with CE method [1]. The application problem of SIS is presented and two MCMC algorithms for application within the SIS procedure are introduced [2]. To reduce the variance of the probability estimator, the adaptive variant algorithms which enhance the performance of subset simulation without additional model evaluation are proposed [3].

In subset simulation, if we apply the parameter to the indicator function, define $\eta_j\left(x;\,\lambda_j\right) := I_g(x;\,\lambda_j)f(x)$, j= 1, ..., T, where $I_g(x;\,\lambda_j)$, j= 1, ..., T is the indicator functions given by

$$I_g\left(x;\,\lambda_j\right) = \begin{cases} 1,\ g(x) \le \lambda_j \\ 0,\ otherwise \end{cases},\ \lambda_1 > \lambda_2 > \cdots > \lambda_T = 0$$

, and $I_g\left(x;\,\lambda_T = 0\right) = I_g\left(x\right)$.

In scaling of PDF, if we apply the parameter to the PDF, define $\eta_j\left(x;\,\lambda_j\right) := I_g(x)f(x;\,\lambda_j)$, j= 1, ..., T, where $f(x;\,\lambda_j)$ is a function defined as follows:

$$f\left(x;\,\lambda_j\right) := \frac{1}{\lambda_j^n} f\left(\frac{x}{\lambda_j}\right),\ \lambda_1 > \lambda_2 > \cdots > \lambda_T = 1$$

The SIS is an adaptive importance sampling method that solves the probability estimation problem using a smooth approximation function with a parameter instead of an indicator function. In SIS, the standard normal CDF $\Phi(\cdot)$ is usually used as an approximation of the indicator function.

This paper use a function $S\left(x\right) = \frac{1}{1+e^{-x}}$ to give an equivalent approximation effect instead of the standard normal CDF as an approximation of the indicator function, with the aim of reducing the simulation computation time.

The trends of the two functions $\Phi(\cdot)$ and $S(\cdot)$ are the same and it is clear that $I_g\left(x\right) = lim_{\lambda\to 0}\, S\left(\frac{-g(x)}{\lambda}\right) = I_g\left(x\right) = lim_{\lambda\to 0}\, \Phi\left(\frac{-g(x)}{\lambda}\right)$.

$\eta_j\left(x;\,\lambda_j\right)$ is defined as follows:

$$\eta_j\left(x;\,\lambda_j\right) := S\left(\frac{-g\left(x\right)}{\lambda_j}\right) f\left(x\right),\ j = 1,\ 2,\ \ldots,\ T$$

Here, $\lambda_1 > \lambda_2 > \cdots > \lambda_T = 0$.

2. Subset Simulation, Scaling of PDF and SIS by one parameter

Since both the final parameters in subset simulation and SIS are $\lambda_T = 0$, $\lambda_T + 1 = 1$ is the final parameter in IS based on the scaling of PDF.

For the three cases, the larger the parameter is, the larger the intermediate failure probabilities are, and so, the estimation problem of failure probability can be solved by simultaneously changing the failure region, the density function and the approximate indicator function by one parameter.

For $\lambda_1 > \lambda_2 > \cdots > \lambda_T = 1$, define $\eta_j\left(x;\,\lambda_j\right)$ as follows:

$$\eta_j\left(x;\,\lambda_j\right) := S\left(\frac{\lambda_j - g\left(x\right)}{\lambda_j}\right) f\left(x;\lambda_j + 1\right),\ j\text{= } 1,\ \ldots,\ T.$$

Then, the estimated value of the failure probability is obtained as follows:

$$\hat{P}_F = P_1 \prod_{j=1}^{T-1} \frac{1}{n_j} \int_{k=1}^{n_j} \frac{S\left(\frac{\lambda_{j+1} - g\left(x_j^k\right)}{\lambda_{j+1}}\right) f\left(x_j^k;\lambda_{j+1} + 1\right)}{S\left(\frac{\lambda_j - g\left(x_j^k\right)}{\lambda_j}\right) f\left(x_j^k;\lambda_j + 1\right)}$$

3. Numerical Experiments

In the reliability problem with the two-dimensional four-branch system function $\varphi\left(x\right)$, the result of estimating $P(g_{12}\left(x\right) < 0)$ by taking a sufficiently large number of samples with a test function $g_{12}\left(x\right) = 12 - \varphi\left(x\right)$ is 1.18×10^{-6} [4]. When the number of samples is 2,000, the RB of the proposed method is 0.18, and the estimator of the proposed method is very close to the actual value.

The result of estimating $P(g_{10}(x) < 0)$ by taking a sufficiently large number of samples with a test function $g_{10}(x) = 10 - \varphi(x)$ is 2.22×10^{-3} [4]. The results show that the effectiveness of the proposed method is not well visible.

For the polynomial square root function $\varphi(x)$, the result of estimating $P(g_6(x) < 0)$ by taking a sufficiently large number of samples with a test function $g_6(x) = 6 - \varphi(x)$ is 2.35×10^{-6} [4]. When the number of samples is 2,500, the relative bias of the proposed method is 0.56, and the estimator is very close to the actual value.

Consequently, the proposed method seems suitable for problems estimating small probabilities of the around 10^{-6}.

The results of the calculation time measurements show that the simulation computation time decreases by one quarter when the newly proposed approximation $S(x)$ of the indicator function is used.

This is expected to be a great help for researchers using low-end computers.

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Spectral-Galerkin Method for Acoustic Transmission Eigenvalue Problems with Anisotropic Media in 3D

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In this papaer, we propose an efficient spectral Galerkin method for acoustic transmission eigenvalue problem with anisotropic media in three dimension. First we reformulate the original problem as a fourth order eigenvalue problem by introducing some auxiliary variables and state a linear weak formulation by introducing an auxiliary function. Then we discuss the spectral Galerkin approximation, and prove the error estimates for both the eigenvalues and eigenfunctions by using the spectral theory of the compact operator and the approximation property of the orthogonal projection operators in non-uniform weighted Sobolev spaces. Numerical examples are reported to validate the effectiveness of the proposed spectral method.

Keywords: transmission eigenvalue, spectral method, anisotropic media, error estimates

In the case of anisotropic media, the transmission eigenvalue problem includes the matrix value function representing the anisotropic property, which makes the variational formulation difficult. The spectral method which has high-accuracy with low computational cost was researched in [2,3] for solving the acoustic transmission eigenvalue problems for isotropic media. We study the spectral Galerkin method for the acoustic wave transmission eigenvalue problem for an anisotropic medium in 3D. In this paper we assume that the sound velocity is the same between inside and outside the scattering object, that is, $n(x)$=1. Then, the acoustic wave transmission eigenvalue problem for an inhomogeneous anisotropic medium in 3D is established as follows [1].

$$\begin{array}{ll} \nabla\cdot A\nabla w+k^2w=0, & x\in D\\ \Delta v+k^2v=0, & x\in D\\ w-v=0, & x\in\partial D\\ \frac{\partial w}{\partial\nu_A}-\frac{\partial v}{\partial\nu}=0. & x\in\partial D. \end{array} \tag{1}$$

Let $\tau:=k^2$, $w=\tau u$, $u:=A\nabla w-\nabla v$, $N:=A^{-1}$, $W=\{u\in H_0(\mathrm{div},D):\nabla\cdot u\in H_0^1(D)\}$, $X=W\times W$. We also define the finite dimensional subspace $X_N:=(W\times W)\cap((P_N(D))^3\times(P_N(D))^3)$. Then the spectral Galerkin approximation scheme of the variational formulation (1) is the following.

$$A((u_N,w_N),(v_N,z_N))=\tau B((u_N,w_N),(v_N,z_N)),\quad \forall(v_N,z_N)\in X_N. \tag{2}$$

Then, the sesquilinear forms A,B is continuous, and A is coercive. Therefore, (2) have solutions and solution operator is existed.

We introduce the d-dimensional weighted Sobolev space $B_{k,l}^m(D):=\{u:\partial_x^s u\in L^2_{\omega^{k+s,l+s}}(D),0<|s|_1\le m\}$ equipped with the following norm and semi-norm

$$\|u\|_{B_{k,l}^m(D)}=\left[\sum\nolimits_{0<|s|_1\le m}\|\partial_x^s u\|^2_{\omega^{k+s,l+s},D}\right]^{1/2},|u|_{B_{k,l}^m(D)}=\left[\sum\nolimits_{j=1}^3\|\partial_x^s u\|^2_{\omega^{k+me_j,l+me_j},D}\right]^{1/2}.$$

Also, we define the spectral projection E,E_N and some distances as follows.

$$\begin{gathered} E=\tfrac{1}{2\pi i}\int R_z(T)\,dz,\quad E_N=\tfrac{1}{2\pi i}\int R_z(T_N)\,dz,\\ d(Q_1,Q_2)=\sup\nolimits_{(\mathbf{u},\mathbf{w})\in Q_1,\|(\mathbf{u},\mathbf{w})\|_X=1}\inf\nolimits_{(\mathbf{v},\mathbf{z})\in Q_2}\|(\mathbf{u},\mathbf{w})-(\mathbf{v},\mathbf{z})\|_X,\\ \delta(R(E),R(E_N))=\max\{d(R(E),R(E_N)),d(R(E_N),R(E))\}, \end{gathered}$$

Then we have the following error estimates of the approximation of the eigenvalues and eigenfunctions.

Theorem 1. *There exists a constant C>0 such that*

$$\delta(R(E),R(E_N))\prec N^{2-m}(|u_1|_{B^m_{k_1,k_1}(D)}+|u_2|_{B^m_{k_2,k_2}(D)}+|u_3|_{B^m_{k_3,k_3}(D)}),$$

$$|\tau-\tau_{j,N}^{-1}|\prec N^{\frac{2(2-m)}{\alpha}}\left[(|u_1|_{B^m_{k_1,k_1}(D)}+|u_2|_{B^m_{k_2,k_2}(D)}+|u_3|_{B^m_{k_3,k_3}(D)})(|u_1^*|_{B^m_{k_1,k_1}(D)}+|u_2^*|_{B^m_{k_2,k_2}(D)}+|u_3^*|_{B^m_{k_3,k_3}(D)})\right]^{1/\alpha}.$$

Theorem 2. *Let $\tau_N=k_N^2$ be an eigenvalue of* (??) *such that $lim_{N\to\infty}\tau_N=\tau$. Suppose for each N that (u_N,w_N) satisfy $\|(u_N,w_N)\|_X=1$and $(\tau_N^{-1}-T)^k(u_N,w_N)=0$ for some positive integer $k\le\alpha$. Then, for any integer l with $k\le l\le\alpha$, there exist a vector (u,w) such that $(\tau^{-1}-T)^l(u,w)=0$ and*

$$\|(u,w)-(u_N,w_N)\|_X\prec(N^{2-m}(|u_1|_{B^m_{k_1,k_1}(D)}+|u_2|_{B^m_{k_2,k_2}(D)}+|u_3|_{B^m_{k_3,k_3}(D)}))^{\frac{l-k+1}{\alpha}}.$$

Now, we present numerical examples. We computed on the unit cube $[0,1]^3$with A as follows:

$$A_1 = \frac{1}{4}I_3, A_2 = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}, A_3 = \begin{bmatrix} 1/4 & -1/8 & 0 \\ -1/8 & 3/8 & -1/8 \\ 0 & -1/8 & 1/4 \end{bmatrix}.$$

The numerical results of the first six real eigenvalues are listed in Table 1. Our numerical results are consistent to the results of the finite element method in [1].

Table 1: The first six real transmission eigenvalues with A_1, A_2, A_3 for N=14.

N	k_1	k_2	k_3	k_4	k_5	k_6
A1	5.2599361	5.9202893	5.9202893	5.9202893	6.3608806	6.3608806
A2	4.7266463	5.067341	5.5605206	5.7195872	6.22882	6.383237
A3	4.8837335	4.9846962	5.8001326	5.8034725	5.9488099	6.2824985

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An Improved YOLO Model for Real-Time Small Object Detection on Aerial Images

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In this paper, we propose an improved algorithm of YOLO for small object detection on aerial images and compare its accuracy and speed.

Keywords: real-time detection, aerial image, coordinate attention module, loss function

1 Proposed method

Unlike traditional loss functions such as CIoU and WIoU based on overlap and distance metrics, ScaleShape-IoU reflects the type, aspect ratio and size of objects in a loss function. This is particularly beneficial for small and irregular shapes in spatial images. Our proposed model, combining an optimal feature pyramid detection head, a coordinate attention mechanism that focuses on coordinate information such as CA, and an innovative ScaleShape-IoU loss function, allows us to achieve real-time and high accuracy for detecting small objects in applications of aircrafts including UAVs.

-Scale-IoU

$$IoU^{Scale} = \frac{S_{inter}}{S_{union}} \tag{1}$$

where S_{inter} is the intersection area and S_{union} is the union area of the transformed boxes(conversion proportionality coefficient is)

- Matching formula for aspect ratio

$$T^{shape} = \left(\frac{h}{c} - \frac{h^{gt}}{c^{gt}}\right)^2 \tag{2}$$

where c is the diagonal length of the ground truth box and c^{gt} is the diagonal length of the prediction case.

- Size matching formula

$$S^{shape} = \sum_{t=w,h} \left(1 - e^{-w_t}\right)^{\theta}, \quad 2 \leq \theta \leq 5 \tag{3}$$

, where

$$\begin{cases} \omega_w = \frac{|w-w^{gt}|}{\max(w,w^{gt})} \\ \omega_h = \frac{|h-h^{gt}|}{\max(h,h^{gt})} \end{cases} \tag{4}$$

The exponent θ is chosen as the optimum value through the experiment.

Finally, the ScaleShape-IoU loss function is constructed by combining the above terms.

$$L_{ScaleShape-IoU} = 1 - IoU^{Scale} + T^{Shape} \times D^{Shpae} + S^{Shape} \tag{5}$$

where D^{Shpae} is normalized distance.

This loss function allows the model to predict boxes that accurately match the center position, size and aspect ratio of objects in aerial images.

By combining the scaling of the predicted bounding box and the matching of aspect ratio and size, this loss function offers several advantages:

Small object detection: Proportional transformation of the bounding box allows the model to better localize small objects, which is very important in aerial image datasets such as VisDron2019 et al.

Shape matching: The matching of aspect ratio and size matching can well reflect the loss of ground truth box and predicted bounding box that are not consistent in aspect ratio and size, thus increasing the convergence speed and accuracy of the model.

This design shows better overall performance, especially when dealing with complex spatial images with diverse object distributions.

2 Experimental and analysis

we have performed and analyzed the benchmark for the dataset VisDrone2019-DET+COCO.

Table 1. Comparison of $AP_{0.5}$ values with the exponential and the conversion proportionality coefficients of the loss function.

	$k = 1.2$	$k = 1.3$	$k = 1.4$	$k = 1.5$
$\theta = 2.0$	46.8	47.4	47.9	47.3
$\theta = 3.0$	47.2	47.8	48.3	47.7
$\theta = 4.0$	47.0	47.3	48.1	47.9
$\theta = 5.0$	46.7	47.2	47.7	47.4

Table 2. Performance comparison

Model	Parameters(M)	$FLOP_s$(G)	$AP_{.50}$(%)
YOLO-7	5.0	22.5	46.6
Proposed model	4.29	20.9	48.3

As shown in Table 1, the most optimal value is $= 1.4$ and $= 3.0$.

Next, we compare our model performance with the previous model (YOLO-7).

Through the training results, we can see that the performance of detecting small objects in aerial images is improved compared to the previous model.

In this paper, we propose an improved YOLO for real-time small object detection in aerial images, and compare its accuracy with the previous YOLO-7 model in the dataset VisDrone2019-DET+COCO, and demonstrate its superiority.

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Modeling of Free Flow in Stamped Tubular Ceramic Membrane Using Finite Element – Lattice Boltzmann Method

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One way to resist fouling in the filtration of fluids containing particulates such as beer yeast suspensions is to modify the inner surface of the membrane. In this paper, the free flow of fluid containing particulates in the flow channel of the stamped membrane which has a helical reversed thread on the inner surface was modeled and numerically analyzed using the finite element-lattice Boltzmann method. The axial cross-section of the flow channel was taken as the solution domain and the finite element method was combined to treat the curved boundary of the solution domain. Numerical experiments confirmed that recirculation existing in the flow channel of the stamped membrane and it resists the formation of cake layer.

Keywords: fouling, membrane, lattice Boltzmann method, finite element method, curve boundary

Several methods such as hydrodynamic operation, surface modification of membranes, and regularly cleaning membranes have been proposed to reduce fouling. J. Stopka et al. [1] experimentally investigated the possibility of permeate flux enhancement based on surface modification of the membrane using a stamped membrane which has a helically stamped inner surface(Fig.1). The authors argued that stamped ceramic membrane has several advantages compared to smooth membrane, such as higher flux at the same velocity of the feed and lower power consumption required for a unit volume of permeate.

Figure 1: Radial and axial cross-sections of the stamped membrane [1].

The fluid flow model consists of the Navier-Stokes equation (NSE) describing the free flow of fluid inside the flow channel, and the advection-diffusion equation (ADE) for particle concentration. The evolution of Lattice Boltzmann equation (LBE) for NSE and ADE with Bhatnagare Grosse Krook (BGK) approximation is described by:

$$\frac{\partial f_i}{\partial t} + c_i \cdot \nabla f_i = -\frac{1}{\tau_f}(f_i - f_i^{eq}) + F_i \Delta t \tag{1}$$

$$\frac{\partial g_i}{\partial t} + c_i \cdot \nabla g_i = -\frac{1}{\tau_g}(g_i - g_i^{eq}) \tag{2}$$

, where f_i and g_i are the distribution functions for NSE, ADE, respectively, with velocity $\boldsymbol{c}_i$ at the lattice node $\boldsymbol{x}$ and time t, t is the time increment, $f_i{}^{eq}$, $g_i{}^{eq}$ are the ith equilibrium distribution functions, and τ_f, τ_g are the collision relaxation times. The source term of Eq. (1) F_i is $F_i = \frac{\Delta t}{1-2\tau_f} w_i F$, where external force F is$F = -\frac{1}{\rho}\nabla p$.

Fluid density ρ and velocity $\boldsymbol{u}$, particle concentration C can be obtained from the distribution functions.

$$\rho = \sum_i f_i \tag{3}$$

$$\rho u = \sum_i f_i c_i \tag{4}$$

$$C = \sum_i g_i \tag{5}$$

The distribution functions in Eqs. (1) and (2) are solved as follows. First, the first terms on the left-hand side of Eqs. (1) and (2) are discretized in time to obtain

$$f_i(x,t+\Delta t) - f_i(x,t) + c_i \cdot \nabla f_i \Delta t = -\frac{\Delta t}{\tau_f}(f_i - f_i^{eq}) + F_i \Delta t \tag{6}$$

$$g_i(x,t+\Delta t) - g_i(x,t) + c_i \cdot \nabla g_i \Delta t = -\frac{\Delta t}{\tau_g}(g_i - g_i^{eq}) \tag{7}$$

By computing the third terms of these equations with the finite element method, we can obtain the value of the distribution function at the next time step at the point of interest.

The boundary conditions are as follows: In the concentration field, the inlet side is set to the non-homogeneous Dirichlet condition, the outlet side to the homogeneous Neumann boundary condition and the wall boundary to the homogeneous Dirichlet condition.

We have performed simulations for three types of cross-sections by using D2Q9model. That is, assuming the curved parts of the boundary (corresponding to the surface of the stamped membrane) in the axial section to be sinusoidal, the upper and lower curves are generated with different periods (We shall call this *a-type cross-section*) or the same period (likewise, *b-type* or *c-type*). In particular, in the case of the same period, the curves are generated either axisymmetrically (*b*-type) or parallel (*c*-type).

The simulation results are obtained at dimensionless time step of $2{\times}10^4$ and here all physical quantities are dimensionless values.

Fig. 2 shows the concentration distribution of particles inside the flow channel.

In the figure, the particle concentrations on the inlet side of the fluid are the same, but the particle concentrations at the outlet side towards membrane exit are different. In Fig. 2a, the particles are relatively well ejected towards the outlet, because the lower part of the cross-section of the flow channel has less pipe resistance than the upper part, so that both the fluid and fine particles are well ejected. Thus, poor fouling, but poor permeate in the bottom surface of the channel. Throughout Fig. 2b, we can hardly see the particles that are drained towards the exit, which indicates that the inner wall of the flow channel is heavily fouling. And it can be seen from Fig. 2c that the particles are more concentrated in the middle part of the flow channel axis. In particular, the highest concentration value is the largest among the three cases, which indicates the highest packing of particles. This is due to the recirculation near the overall surface, which prevents particles from approaching the wall of the flow channel and pushes it out into the middle of the channel.

Overall, the worst membrane in the three cases is the one with a *b*-type cross-section and the best membrane is the one with a *c*-type cross-section, with good permeate and low fouling. There exists recirculation when the fluid flows freely inside the flow channel of the stamped membrane, and it resists fouling.

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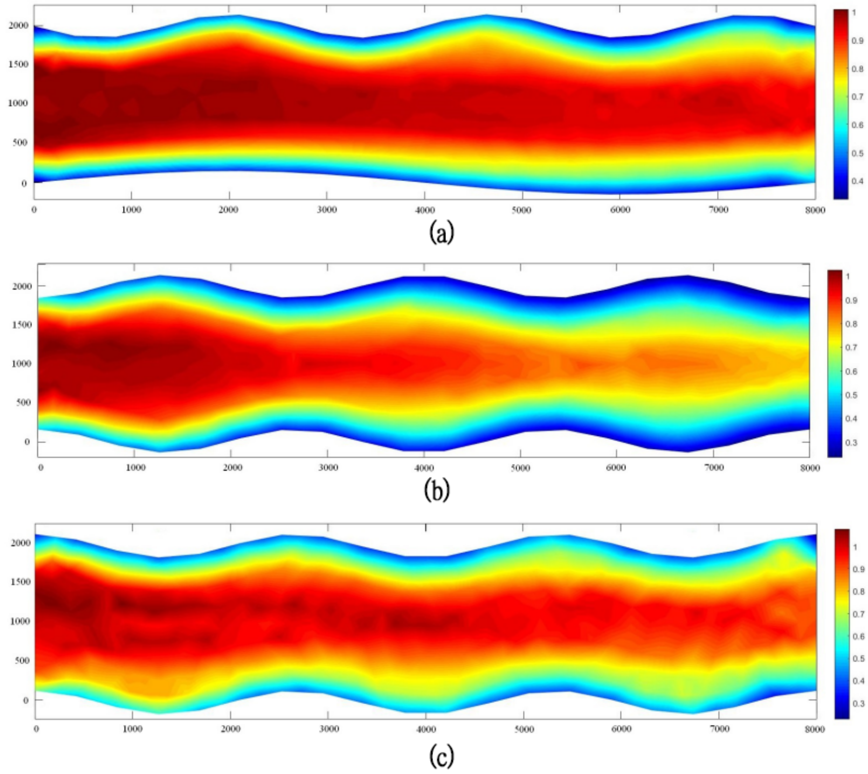

Figure 2: Particle concentration in the axial cross-sections of three types flow channels (a) a-type, (b) b-type and (c) c-type

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A Problem of Identifying Input Signals of Dynamic systems Modelled by Volterra Polynomials*

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The paper considers an approach to identifying input signals in models based on Volterra polynomials. The model structure is predetermined, taking into account the domain in the space of input and output parameters. It is assumed that the input signal $x(t,l) = (x_1(t,l), x_2(t,l))^T$ and the response $y(t,l) = (y_1(t,l), y_2(t,l))^T$ are vector functions, where t is time and l is the amplitude (height) of the test signals used to construct the Volterra polynomial.

Keywords: nonlinear dynamical systems, integral equations, Volterra polynomial

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1 Introduction

A significant number of dynamic systems (technical, physical, biological) are nonlinear, while their full-scale modeling is either difficult or impossible [1]. In many applications, nonlinear dynamics is described by the tool of Volterra integro-power series [2], which, due to its universality, represents a promising direction in mathematical modeling. Problems leading to the identification of nonlinear dynamic systems are associated with the problem of Volterra kernels reconstruction [3] and determination of input signals [4] that provide a desired (specified) response. Most often, identification methods are considered with respect to stationary processes. As shown in [5], accounting for nonstationary properties of a dynamic system makes it possible to increase modeling accuracy by obtaining more complete information about the system's responses to test inputs. In [6], the problem of kernel identification in modified Volterra polynomials

$$y(t,l)=\sum_{m=1}^{N}\int_0^t\dots\int_0^t\int_D K_m(t,l,\tau_1,\dots,\tau_m,s)\prod_{i=1}^{m}x(\tau_i,s)d\tau_i ds \tag{1}$$

for describing a nonstationary nonlinear dynamic system with distributed parametersis considered. In (1), $t\in[0,T]$ is time; $D\subset\mathbb{R}^n$ is the domain in space where the input and output signals with amplitude $l=(l_1,l_2,\dots,l_n)\in D$ are observed; the Volterra kernels are symmetric with respect to the variables τ_1, ..., τ_m.

The aim of this paper is to consider a new class of polynomial Volterra integral equations of the first kind, to the solution of which the problem of identifying $x(t,l)$ in (1) can be reduced.

2 Main results

Let $l\in[0,L]$ be the amplitude of the input signal $x(t,l)=(x_1(t,l),x_2(t,l))^T$ from the admissible class of test signals used in the problem of Volterra kernels identification. Consider the problem of identifying the disturbance $x(t,l)$ that provides a desired response $y(t,l)=(y_1(t,l),y_2(t,l))^T$ for given transient characteristics (Volterra kernels). Assuming that the dynamic system has quadratic nonlinearity ($N=2$), we pass from (1) to a system of polynomial Volterra integral equations of the first kind:

$$\int_0^t\int_0^l\begin{pmatrix}K_1^{(1)}(t,l,\tau,s) & K_2^{(1)}(t,l,\tau,s)\\ K_1^{(2)}(t,l,\tau,s) & K_2^{(2)}(t,l,\tau,s)\end{pmatrix}\begin{pmatrix}x_1(\tau,s)\\ x_2(\tau,s)\end{pmatrix}d\tau ds+ \tag{2}$$

$$+\int_0^t\int_0^t\int_0^l\begin{pmatrix}K_{11}^{(1)}(t,l,\tau_1,\tau_2,s) & \frac{1}{2}K_{12}^{(1)}(t,l,\tau_1,\tau_2,s) & \frac{1}{2}K_{12}^{(1)}(t,l,\tau_2,\tau_1,s) & K_{22}^{(1)}(t,l,\tau_1,\tau_2,s)\\ K_{11}^{(2)}(t,l,\tau_1,\tau_2,s) & \frac{1}{2}K_{12}^{(2)}(t,l,\tau_1,\tau_2,s) & \frac{1}{2}K_{12}^{(2)}(t,l,\tau_2,\tau_1,s) & K_{22}^{(2)}(t,l,\tau_1,\tau_2,s)\end{pmatrix}\times$$

$$\times\left[\begin{pmatrix}x_1(\tau_1,s)\\ x_2(\tau_1,s)\end{pmatrix}\otimes\begin{pmatrix}x_1(\tau_2,s)\\ x_2(\tau_2,s)\end{pmatrix}\right]d\tau_1 d\tau_2 ds=\begin{pmatrix}y_1(t,l)\\ y_2(t,l)\end{pmatrix},$$

where the Volterra kernels $K_{12}^{(i)}$ are asymmetric with respect to the arguments τ_1,τ_2 ($i=1,2$). In (2), it is taken into account that:

$$\int_0^t\int_0^t K_{12}^{(i)}(\cdot,\tau_1,\tau_2,s)x_1(\tau_1,s)x_2(\tau_2,s)d\tau_1 d\tau_2 ds=\int_0^t\int_0^t K_{12}^{(i)}(\cdot,\tau_2,\tau_1,s)x_1(\tau_2,s)x_2(\tau_1,s)d\tau_1 d\tau_2 ds.$$

As noted in [4], the system of equations (2) can be regarded as a system of linear equations with a perturbed right-hand side. The specifics of linear two-dimensional systems equivalent to a system of integral equations of the second kind with an identically degenerate matrix in front of the principal part are considered in [7].

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On the Periodic Problem for Some Classes of Differential Variational Inequalities

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In this paper, using topological methods of the theory of multivalued mappings, we study the problem of the existence of periodic solutions for some classes of differential variational inequalities.

Keywords: differential variational inequalities, topological methods, periodic solutions

1 The main results

The concept of differential variational inequalities (DVI) was introduced by Aubin J.-P. and Cellina A. [1]. However, the first systematic study of DVI was undertaken by Pang J.-S. and Stewart D. E. [2]. DVIs are useful for representing models incorporating both dynamics and inequality constraints, which arise in many applied problems, such as shock interaction problems in mechanics, electrical circuits with ideal diodes, Coulomb friction problems for contacting bodies, economic dynamics, and related problems such as dynamic transportation networks. Some results on the existence of solutions for DVIs can be found in [2]– [4].

In this paper, using topological methods from the theory of multivalued maps, we study the periodic problem for a class of DVIs considered for the initial-value problem in [2], [5], [6]. The motivation for studying this class of DVIs stems from many applications, including linear complementary systems, differential complementary problems and evolutionary variational inequalities.

Let $I = [0, T]$; $C(I, \mathbb{R}^n)[L^2(I, \mathbb{R}^n)]$ denote the spaces of all continuous [resp., summable] functions $u : I \to \mathbb{R}^n$ with the usual norms. By $W_T^{1,2}(I, \mathbb{R}^n)$ we denote the space of Sobolev functions satisfying the periodicity condition $x(0) = x(T)$. For $\tau > 0$, we denote by $\mathcal{C}$ the space $C([-\tau, 0]; \mathbb{R}^n)$. For a function $x(\cdot) \in C([-\tau, T]; \mathbb{R}^n)$, $T > 0$, we denote by $x_t \in \mathcal{C}$ the function defined as $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. Let $K \subset \mathbb{R}^m$ be a non-empty closed convex subset. Let us consider the differential variational inequality of the form:

$$\begin{cases} x'(t) \in \lambda A x(t) + F(t, x_t) + B(t, x_t)u(t) \quad \text{a.e.} \quad t \in I,\ \lambda > 0; \\ \langle \tilde{u} - u(t), G(t, x_t) + Q(u(t)) \rangle \geq 0 \quad \text{a.e.} \quad t \in I, \forall \tilde{u} \in K; \\ x(0) = x(T), u(t) \in K; \end{cases} \tag{1}$$

(A) $A : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded linear operator such that

$$\langle Az, z \rangle \geq a||z||^2 \quad \text{for all } z \in \mathbb{R}^n;$$

(F) $F : I \times \mathcal{C} \to Kv(\mathbb{R}^n)$ satisfies the following conditions:
(i) for each $\varphi \in \mathcal{C}$ multifunction $F(\cdot, \varphi) : I \to \mathbb{R}^n$ admits a measurable selection;
(ii) for a.e. $t \in I$ multimap $F(t, \cdot) : \mathcal{C} \to \mathbb{R}^n$ is upper semicontinuous;
(iii) for every $r > 0$ there exists a positive number $\alpha_{r,F}$ such that for every $\varphi \in \mathcal{C}, ||\varphi|| \leq r$,

$$||F(t, \varphi)|| := \sup\{||y|| : y \in F(t, \varphi)\} \leq \alpha_{r,F}(1 + ||\varphi||) \text{ a.e. } t \in I;$$

(B) $B : I \times \mathcal{C} \to \mathbb{R}^{n \times m}$ is a continuous map such that for every $r > 0$ there exists a positive number $\alpha_{r,B}$ such that for every $\varphi \in \mathcal{C}$ with $||\varphi|| \leq r$

$$||B(t, \varphi)|| \leq \alpha_{r,B} \text{ a.e. } t \in I;$$

(G) $G : I \times \mathcal{C} \to \mathbb{R}^m$ is a continuous map such that for every $r > 0$ there exists a positive number $\alpha_{r,G}$ such that for every $\varphi \in \mathcal{C}$ with $||\varphi|| \leq r$

$$||G(t, \varphi)|| \leq \alpha_{r,G}(1 + ||\varphi||) \text{ a.e. } t \in I;$$

(Q) $Q : \mathbb{R}^m \to \mathbb{R}^m$ is a monotone on K, i.e.

$$\langle u - \tilde{u}, Q(u) - Q(\tilde{u}) \rangle \geq 0 \text{ for all } u, \tilde{u} \in K;$$

there exists $\nu_0 \in K$ such that

$$\lim_{\nu \in K, ||\nu|| \to \propto} \frac{\langle \nu - \nu_0, Q(\nu) \rangle}{||\nu||^2} > 0.$$

Remark 1. From conditions (F) it follows that the superposition multioperator $\mathfrak{P}_F : C([-\tau, T]; \mathbb{R}^n) \to P(L^2(I; \mathbb{R}^n))$ associating each function $x(\cdot)$ the set of all summable selections of the multifunction $F(t, x_t)$, i.e. $\mathfrak{P}_F(x) = \{f \in L^2(I; \mathbb{R}^n) : f(s) \in F(s, x_s) \text{ a.e. } t \in I\}$ is well defined (see, e.g., [7]).

By a solution of problem (1) we mean a function $x \in W_T^{1,2}(I, \mathbb{R}^n)$, for which there exists a function $f \in \mathfrak{P}_F(x)$ and an integrable function $u : I \to K$ such that

$$\begin{cases} x'(t) = \lambda A x(t) + f(t) + B(t, x_t)u(t) & \text{a.e. } t \in I, \\ \langle \tilde{u} - u(t), G(t, x_t) + Q(u(t)) \rangle \geq 0 & \text{a.e. } t \in I, \forall \tilde{u} \in K. \end{cases}$$

Theorem 1. *Let conditions* $(A), (F), (B), (G)$ *and* (Q) *be satisfied. Then problem (1) has a solution for each*

$$\lambda > (\rho\alpha_{r,B}\alpha_{r,G} + \alpha_{r,F})\frac{1}{a}.$$

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Applied Problems of Space Dynamics

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This paper is dedicated to the scientific legacy of Viktor Sergeevich Novoselov, Professor Emeritus of Saint Petersburg State University, Honored Scientist of the Russian Federation, and Doctor of Physics and Mathematics, whose centenary is celebrated this year. The study examines key research directions developed by Professor Novoselov and the successors of his scientific school. In particular, the work highlights contributions to the modeling and solution of space dynamics problems: the motion of mechanical systems with variable mass, optimization of space trajectories, and the stabilization of artificial space objects near their equilibrium positions. Furthermore, the motion of dynamical systems under random perturbations and forces is considered. The paper summarizes the fundamental results of these theoretical investigations and provides the formulations and outcomes of various applied problems in space dynamics.

Keywords: analytical mechanics, space dynamics, controlled systems, robotics, biomechanics, quantum mechanics, statistical physics

1 The main results

This year marks the centenary of the birth of Viktor Sergeevich Novoselov (1926–2019) — an outstanding scientist, Professor Emeritus of Saint Petersburg State University, Honored Scientist of the Russian Federation, Doctor of Physics and Mathematics, and the founder of a scientific school in analytical mechanics, space dynamics, biomechanics, and applied mathematics.

His research interests were remarkably diverse and, alongside the work of his successors, encompass a vast class of theoretical and applied problems. This paper considers only those results of his activity that directly relate to the solution of space dynamics problems.

One of the significant subjects of V.S. Novoselov's research was the mechanical system of variable mass. This is defined as a system of particles with constant masses whose composition changes over time. Within this framework, several novel results were obtained: Meshchersky's equation was refined, Zhukovsky's equation for the motion of a variable-mass body was generalized, and solutions were presented for Tsiolkovsky's problems regarding the motion of a variable-mass point under the influence of a uniform or central gravitational field. Furthermore, a justification was provided for the problems of analytical mechanics of variable-mass systems, accounting for the internal displacement of system particles or bodies. These and other results are detailed in the works [1,2].

Building on the works of Chetaev, V. S. Novoselov developed a general approach to the problem of varying generalized velocities [3–5]. The resulting fundamental achievement was the generalization of the Hamilton-Ostrogradsky principle of least action. Using variational techniques, several remarkable results were obtained for the dynamics of controlled systems. In particular, work [6] proposed and repeatedly applied a general scheme for constructing analytical approximations to solve equations describing the necessary conditions for optimizing impulsive space trajectories in a gravitational field.

Furthermore, work [3] presents fundamental results on the motion of nonholonomic mechanical systems, including nonlinear ones. Novoselov introduced the novel concept of nonlinear nonholonomic coordinates, which allowed for the derivation of the most general form of the equations of motion for nonholonomic systems with nonlinear and non-stationary constraints.

A significant class of Viktor Sergeevich's research involves stochastic controlled systems, which are characterized by accounting for the influence of random factors in control execution, state estimation, and the formulation of the dynamic model operator or phase variable selection conditions. Works [7–9] describe the development of the theory of system motion under the influence of random perturbations or random forces.

While the primary direction of V. S. Novoselov's research was related to optimal control problems in space dynamics, his research topics and methods continually expanded [10,11]. V. S. Novoselov's persistent interest in challenging or novel problems of analytical mechanics, motion control of mechanical systems, and space dynamics defined new areas of study: robotics, quantum mechanics and statistical physics, biomechanics of living systems, and neurodynamics.

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On the Electro-Hydrodynamical Boundary Value Problem for the Unit Cell of Cation-Exchange Membrane

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We study a model problem on the filtration of a conducting fluid through a porous layer. A porous medium is presented as an assemblage of identical spherical cells. Each cell consists of a porous core and liquid shell. We derive apriori estimates for flow characteristics which show the specific behavior of the fluid. Our estimates are validated numerically.

Keywords: fluid flow, porous medium, weak solution, Debye radius

We study the following system of differential equations which describes our filtration process through a porous charched membrane:

$$
\begin{aligned}
&\nabla p^{\kappa} = m_{\kappa}\Delta \mathbf{v}^{\kappa} - \left(Z_+ C_+^{\kappa} - Z_- C_-^{\kappa}\right)\nabla\varphi^{\kappa} - \mu_{\kappa}\mathbf{v}^{\kappa},\\
&\nabla\cdot\mathbf{v}^{\kappa} = 0,\\
&\delta^2\Delta\varphi^{\kappa} = -\left(Z_+ C_+^{\kappa} - Z_- C_-^{\kappa} - \sigma_{\kappa}\right)\\
&\nabla\cdot\mathbf{j}_{\pm}^{\kappa} = 0,\\
&\mathbf{j}_{\pm}^{\kappa} = \mathbf{v}^{\kappa} C_{\pm}^{\kappa} - \frac{1}{\nu^{\kappa}\mathrm{Pe}}\left(\nabla C_{\pm}^{\kappa} \pm Z_{\pm} C_{\pm}^{\kappa}\nabla\varphi^{\kappa}\right).
\end{aligned}
\qquad (1)
$$

Here p^{κ} is local pressure, $\mathbf{v}^{\kappa}$ is velocity vector, μ^{κ} is a dynamic viscosity, φ^{κ} is the electric potential, $Z_{\pm}$ are charge modules of cations and anions of the electrolyte, $C_{\pm}^{\kappa}$ are concentrations of cations and anions, $\mathbf{j}_{\pm}^{\kappa}$ are ions flux densities. The system (1) is considered in the unit cell of the memrane. The continuity boundary conditions are assumed to be valid on the boundaries of the partcles.

We derive estimates for flow characteristics and validate it numerically for different sets of fliud parameters.

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Classical Solution to the Second Mixed Problem for a Semilinear Wave Equation with a Dirac Potential

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For a semilinear wave equation with a Dirac potential, we consider the solvability and uniqueness of the second mixed problem in the first quadrant of the plane.

Solution to the problem is built using the method of characteristics in an implicit analytical form as solution to coupled integral equations. We study the solvability of these equations depending on the initial data and their smoothness. For the problems under consideration, the uniqueness of the solutions is proven and the conditions under which classical solutions exist are established.

Keywords: semilinear wave equation, classical solution, mixed problem, Dirac potential, matching conditions

In the present paper, we shall consider the question of the global solvability of the second mixed problem

$$\begin{aligned} &\Box_a u(t,x) - \Theta(t,x)\delta_{(t_0,x_0)}[u](t,x) = f(t,x,u(t,x)),\ \ (t,x) \in Q := (0,\infty)\times(0,\infty),\\ &u(0,x) = \varphi(x),\ \ \partial_t u(0,x) = \psi(x),\ \ x \in [0,\infty), \\ &\partial_x u(t,0) = \mu(t),\ \ t \in [0,\infty), \end{aligned} \tag{1}$$

where $\Box_a = \partial_t^2 - a^2\partial_x^2$ is the d'Alembert operator ($a > 0$ for definiteness), (t_0, x_0) is a point lying in the set Q, and $\delta_{(t_0,x_0)}$ is the Dirac delta distribution concentrated at the point (t_0, x_0), i.e., $\delta_{(t_0,x_0)}[u](t,x) = u(t_0,x_0)$.

In the case of an unloaded equation, i.e. $\Theta \equiv 0$, we considered problem (1) in our paper [1], [2]. Similar problems for linear parabolic and hyperbolic equations with a Dirac potential were also solved [3] – [8]. The Cauchy problem in the upper half-plane for the equation

$$\Box_a u(t,x) - \Theta(t,x)\delta_{(t_0,x_0)}[u](t,x) = f(t,x,u(t,x))$$

was solved in our paper [9].

We introduce the notation

$$\Delta(t_P,x_P) = \{(t,x) \mid 0 \le t \le t_P \wedge |x - x_P| \le a|t - t_P|\}$$

and

$$\widetilde{\Delta}(t_P,x_P) = \{(t,x) \mid 0 \le t \le t_P \wedge |x - x_P| \le a|t - t_P| \wedge x \ge 0\}.$$

We denote

$$\bullet_{\text{ext}}(t,x) = \begin{cases} \bullet(t,x), & x \ge 0, \\ \bullet(t,-x), & x < 0. \end{cases}$$

We state the main results of this paper as the following statement.

Theorem 1. *Let the smoothness conditions*

$$\begin{aligned} &f \in C^1(\overline{Q}\times\mathbb{R}),\ \ \varphi \in C^2([0,\infty)),\\ &\psi \in C^1([0,\infty)),\ \ \mu \in C^1([0,\infty)),\ \ \Theta \in C(\overline{Q}),\\ &\left(\overline{Q} \ni (t,x) \mapsto \int\limits_0^t \Theta(\tau, |x \pm a(t-\tau)|)\,d\tau\right) \in C^1(\overline{Q}), \end{aligned}$$

and one of the following conditions be satisfied:

1. The function f satisfies the Lipschitz condition

$$|f(t,x,z_1) - f(t,x,z_2)| \le k(t,x)|z_1 - z_2|,$$

where $k \in L^2_{\mathrm{loc}}(\overline{Q})$ and the smallness condition

$$\iint\limits_{\Delta(t_0,x_0)} \left(|\Theta_{\mathrm{ext}}(\tau,\xi)| + |k_{\mathrm{ext}}(\tau,\xi)|\right) d\tau d\xi < 2a$$

is satisfied.

2. The functions f and Θ satisfy the sign conditions $\partial_u f(t,x,u) \geq 0$ and $\Theta(t,x) \leq 0$ for all $(t,x) \in \overline{Q}$ and the growth condition $|f(t,x,u)| \leq \beta(t,x)|u|^\alpha$, where $\beta \in L^1_{\mathrm{loc}}(\overline{Q})$ and $\alpha \in [0,1)$.

Then the second mixed problem (1) *has a unique classical solution in the class $C^2(\overline{Q})$ if and only if the matching conditions $\mu(0) = \varphi'(0)$ and $\mu'(0) = \psi'(0)$ are satisfied.*

Note that the conditions $\partial_u f(t,x,u) \geq 0$ and $|f(t,x,u)| \leq \beta(t,x)|u|^\alpha$, $\alpha \in [0,1)$, are simultaneously satisfied for functions like

$$f(u) = \arctan(u^{2k+1}), \quad f(u) = \tanh(u^{2k+1}), \quad f(u) = \mathrm{gd}(u^{2k+1}),$$

where $k \in \mathbb{N}$.

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Multidimensional Exact Solutions for a Parabolic Monge–Ampère Equation with Delay*

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A multidimensional Monge–Ampère equation with delay is studied. Families of exact solutions are constructed and expressed explicitly in terms of elementary and special functions.

Keywords: multidimensional Monge–Ampère equation, delay, exact solutions

Numerous models and problems in biology, chemistry, economics, and other fields are formulated in terms of delay partial differential equations (DPDEs), where process dynamics depend not only on the current state but also on the history of the system. The inclusion of delay terms significantly increases the complexity of analyzing such mathematical models and, consequently, hinders the derivation of exact solutions for the underlying equations. Several fundamental results in this area have been established in [1,2,3,4,5]. It is worth noting that the literature primarily focuses on delay equations with a single spatial variable. A comprehensive review of current results and an extensive bibliography, including over 600 references on exact solutions for DPDEs, can be found in monograph [5].

In this paper, we consider a multidimensional parabolic Monge–Ampère equation with delay of the following form:

$$\frac{\partial u}{\partial t} = \det H(u) + \alpha \bar{u}, \tag{1}$$

where $u = u(\mathbf{x}, t)$, $\bar{u} = u(\mathbf{x}, \bar{t})$, $\bar{t} = t - \tau$, $t \geq 0$, $\tau > 0$, $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$; $H(u) = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$ is the Hessian matrix of order n, and $\alpha \neq 0$ is a constant. In the particular case where $n = 2$ and the linear delay term is absent ($\alpha = 0$), equation (1) reduces to the parabolic Monge – Ampère equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2, \quad u = u(x, y, t). \tag{2}$$

This equation arises in magnetohydrodynamics to describe the evolution of electron vortices in magnetized plasmas [6,7].

Multidimensional exact solutions to equation (1) are sought in the form of generalized separation of variables [8]

$$u(\mathbf{x}, t) = \psi(t)F(\xi) + \theta(t), \tag{3}$$

where the argument $\xi = \xi(\mathbf{x})$ of the unknown function F is a quadratic form:

$$\xi = \frac{1}{2}(A\mathbf{x}, \mathbf{x}). \tag{4}$$

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Here, A is a non-zero symmetric numerical matrix of size $n \times n$. Previously, the quadratic function (4) was successfully employed by the authors to construct multidimensional exact solutions for the generalized Monge – Ampère equation [9], a system of equations involving the Monge–Ampère operator and the hyperbolic Monge – Ampère equation.

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To the Control Reconstruction Problem for Systems Non-linear in Controls

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The control reconstruction problem for dynamic systems is considered in this work. That is the problem of finding an unknown control that generated an observed trajectory of a dynamic system. Inaccurate measurements of this trajectory are known. In this work, systems, non-linear in controls, are considered. The definition of the solution is proposed and the problem is stated. An approach for solving the problem is suggested.

Keywords: non-linear systems, control reconstruction, inverse problems

1 Introduction

The control reconstruction problem for systems, linear in controls, was studied in many works. See, for example, the survey article [1] on a family of methods for solving this problem, based on the extremal aiming procedure. The authors of this work proposed another approach, based on auxiliary variational problems [2]. The control reconstruction problem for systems, non-linear in controls, is not studied so well.

2 Input data

The following systems are considered

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)), \quad x(\cdot) : [0, T] \to \mathbb{R}^n, \quad u(\cdot) : [0, T] \to \mathbb{R}^m, \quad t \in [0, T], \quad T < \infty. \tag{1}$$

Let the function $f(t, x, u)$ be continuous in t and u and Lipschitz continuous in x.

The controls $u(\cdot)$ are measurable functions, satisfying the geometric constraints

$$u(t) \text{ a. e. on } [0, T] \ \in \mathbf{U}, \tag{2}$$

where $\mathbf{U} \subset \mathbb{R}^m$ is a compact. Generally speaking, it is non-convex.

It is supposed that some trajectory $x^*(\cdot)$ of system (1) is observed. It is generated by some unknown control, which must be reconstructed by known inaccurate discrete measurements of the observed trajectory y_i^δ:

$$\|y_i^\delta - x^*(t_i)\| \leq \delta, \quad t_i = ih, \quad T = Nh, \quad i = 0, \dots, N. \tag{3}$$

3 The main results

A definition of solution for the control reconstruction problem is formulated. It is a so-called normal control that generates the observed trajectory of the system and is uniquely determined. The justification of the problem statement is based on the theory of differential inclusions (see, for example, [3]).

The task is to construct such approximations of the normal control, based on the measurement sets (3), that satisfy the constraints (2) and converge to the normal control when $\delta, h \to 0$.

It is shown that the problem should be stated different for different types on the control constraints (2). For example, if the constraints are convex, the approximations of the solution can converge strongly in L^2, while in the case of non-convex constraints it is not suitable, and weak convergence in L^2 should be considered.

An approach to solving the stated control reconstruction problem is suggested for the case of convex constraints (2).

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Quasi-steady Vortex States as the Semiclassical Stage of the Vortex Lattice Formation in an Open Nonlinear Quantum System

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We propose the method for constructing asymptotic solutions to the nonlinear Schrödinger equation with non-Hermitian terms that are semiclassically localized on an evolving curve. It is shown that the formation of the vortex lattice within the framework of a certain model in open quantum system has the semiclassical stage that is described by the solutions constructed. This stage is treated as quasi-steady vortex states and is endowed with the geometrical interpretation within the proposed formalism.

Keywords: semiclassical approximation, nonlinear Schrödinger equation, Maslov complex germ method

The formation of vortex lattices in quantum systems is quite complicated for the mathematical description since it is described by solutions with nontrivial geometry of localization and requires taking into consideration the openness of the system due to the dissipativity of the process. Thus, these aspects make the problem of the modelling complex since the minimal model is based on the solutions to the nonlinear Schrödinger equation with an anti-Hermitian term that have a spatially distributed localization domain. We showed that such process admits the semiclassical stage that can be described using our original approach within the framework of the semiclassical approximation.

We consider the Cauchy problem for the equation of the form

$$\begin{gathered}
\Big\{ -i\hbar\partial_t + H(\hat{z},t)[\Psi] - i\hbar\Lambda\breve{H}(\hat{z},t)[\Psi]\Big\}\Psi(\vec{x},t) = 0, \quad \Psi(\vec{x},0) = \psi(\vec{x}),\\
H(\hat{z},t)[\Psi] = V(\hat{z},t) + \kappa\int\limits_{\mathbb{R}^n} d\vec{y}\,\Psi^*(\vec{y},t)W(\hat{z},\hat{w},t)\Psi(\vec{y},t),\\
\breve{H}(\hat{z},t)[\Psi] = \breve{V}(\hat{z},t) + \kappa\int\limits_{\mathbb{R}^n} d\vec{y}\,\Psi^*(\vec{y},t)\breve{W}(\hat{z},\hat{w},t)\Psi(\vec{y},t),
\end{gathered} \tag{1}$$

where $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\hat{\vec{p}} = -i\hbar\partial_x$, $\hat{\vec{p}}_y = -i\hbar\partial_y$, $\hat{z} = (\hat{\vec{p}}, \vec{x})$, $\hat{w} = (\hat{\vec{p}}_y, \vec{y})$, and $\hbar$ is the formal small parameter of the semiclassical approximation. The operators $V(\hat{z}, t)$, $\breve{V}(\hat{z}, t)$, $W(\hat{z}, \hat{w}, t)$, and $\breve{W}(\hat{z}, \hat{w}, t)$ are pseudo-differential operators with smooth symbols.

In [1], we have constructed the solutions to (1) that are semiclassically localized in a neighbourhood of the evolving curve $\Lambda_t = \Big\{ z = Z(s,t) \Big| s \in [s_1, s_2] \Big\}$ in the phase space. Such solutions were obtained using the ideas of the method of semiclassically concentrated states that is based on the Maslov complex germ method [3,2]. To apply such method, we lifted the original problem to the space of variables with the higher dimension. Solutions to the original Cauchy problem are the projection of the solutions constructed in the extended dimension to the original space.

As the basic model of the vortex lattice formation, we study the transient process described by the solutions to the Cauchy problem for the following equation

$$\begin{aligned} &\bigg\{ \frac{-i\hbar\partial_t}{1 - i\hbar\Lambda} + \hat{\vec{p}}^{\,2} + \langle \vec{x}, K\vec{x} \rangle + k_4 |\vec{x}|^4 - \omega(\hat{p}_1 x_2 - \hat{p}_2 x_1) + \\ &+ \frac{\kappa}{\pi\gamma^2} \int\limits_{\mathbb{R}^2} \exp\bigg[-\frac{(\vec{x} - \vec{y})^2}{\gamma^2} \bigg] |\Psi(\vec{y}, t)|^2 d\vec{y} \bigg\} \Psi(x, t) = 0, \end{aligned} \tag{2}$$

where ω corresponds to the rotation rate of the reference frame.

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Controllability of Abstract State-Dependent Delay Systems with Nonlocal Conditions*

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This work presents a study on controllability of a class of abstract semilinear control systems with state-dependent delay and nonlocal condition in Hilbert spaces. The function representing the state-dependent delay shows non-Lipschitz behaviour. Therefore, a new function space of Lipschitz functions is defined with suitable norm and seminorm to implement the framework of strongly continuous

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semigroup of linear operators. Sufficient conditions on the control operator and the nonlinear term have been provided to established the trajectory controllability of the considered system. The main result is obtained by the fixed point approach.

Keywords: trajectory controllability, state-dependent delay, semilinear control systems, nonlocal condition

1 The main results

Let us consider Hilbert spaces $(X, \|\cdot\|)$ and $(U, \|\cdot\|_U)$ of state and control variables, respectively, and denote $\mathcal{C}_t = C([-\tau, t]; X)$ with norm $\|x\|_{\mathcal{C}_t} = \sup_{s\in[-\tau,t]} \|x(s)\|$. Consider the abstract semilinear state-dependent delay control system with nonlocal condition as

$$\dot{x}(t) = Ax(t) + Bu(t) + f(t, x_{\sigma(x_t)}),\ t \in [0, T], \tag{1}$$

$$g(x_0) = \phi \ \text{ on } [-\tau, 0], \tag{2}$$

where $\tau > 0$, $x \in C([-\tau, T]; X) = \mathcal{C}_T$ is state variable, $u \in L^2([0, T]; U)$ is control variable, $A : D(A) \subset X \to X$ is densely defined closed linear operator, $B : U \to X$ is bounded linear operator, $x_t \in C([-\tau, 0]; X) = \mathcal{C}_0$ is delay function, $f : [0, T] \times \mathcal{C}_0 \to X$ is nonlinear function, $\sigma : \mathcal{C}_0 \to [0, T]$ and $g : \mathcal{C}_0 \to \mathcal{C}_0$ are continuous functions, and $x_0, \phi \in \mathcal{C}_0$. The equation (2) represents the nonlocal delay condition.

This work is motivated by the pioneer works of Chalishajar *et al.* [1], Hernandez [2] and Hernandez *et al.* [3].

Assumption 1 *The system operator A is the generator an analytic semigroup $\{S(t)\}_{t\geq 0}$ with $||S(t)|| \leq Me^{\omega t}$ for $M \geq 1$ and $\omega > 0$.*

Let us define a normed linear space

$$C_{Lip}([0, T] \times \mathcal{C}_0; X) = \{\eta : [0, T] \times \mathcal{C}_0 \to X \mid \eta \text{ is Lipschitz function}\}$$

with norm

$$\|\eta\|_{C_{Lip}([0,T]\times\mathcal{C}_0;X)} = \|\eta\|_{C([0,T]\times\mathcal{C}_0;X)} + [\eta]_{C_{Lip}([0,T]\times\mathcal{C}_0;X)},$$

where $[\cdot]_{C_{Lip}([0,T]\times\mathcal{C}_0;X)}$ is the seminorm defined by

$$[\eta]_{C_{Lip}([0,T]\times\mathcal{C}_0;X)} = \sup_{t,s\in[0,T];x,y\in X;t\neq s;x\neq y} \frac{\|\eta(t, x) - \eta(t, y)\|}{|t - s| + \|x - y\|_{\mathcal{C}_0}}.$$

Assumption 2 *$\sigma \in C_{Lip}(\mathcal{C}_0; [0, T])$ with Lipschitz constant $L_\sigma > 0$ and*

$$|\sigma(x_t) - \sigma(y_t)| \leq L_\sigma \|x_t - y_t\|_{\mathcal{C}_0}, \quad t \in [0, T].$$

Assumption 3 *f is continuous in $[0, T]$ and Lipschitz in $\mathcal{C}_0$ with constant $L_f > 0$ satisfying*

$$\|f(t, x_{\sigma(x_t)}) - f(t, y_{\sigma(y_t)})\| \leq L_f(1 + [\sigma]_{C_{Lip}(\mathcal{C}_0;[0,T])})\|x - y\|_{\mathcal{C}_T}.$$

The system (1), (2) has unique mild solution $x \in C([-\tau, T]; X)$ under Assumptions (1)-(3) given by $x|_{[-\tau,0]} = \phi$ and the integral equation

$$x(t) = S(t)\phi(0) + \int_0^t S(t - s)Bu(s)ds + \int_0^t f(s, x_{\sigma(x_s)})ds, \quad t > 0. \tag{3}$$

Let us take $\mathfrak{T}$ as the collection of $y \in C([0, T]; X)$ such that $y(0) = \phi(0)$, $y(T) = x_1$ and y is differentiable almost everywhere.

Definition 1 *[1] We say that the system* (1), (2) *is trajectory controllable ($\mathfrak{T}$-controllable) if for every $y \in \mathfrak{T}$, there exists a control $u \in L^2([0,T];U$ such that the corresponding solution $x(\cdot)$ satisfies $x(t) = y(t)$ almost everywhere.*

Assumption 4 *B satisfies $\langle Bu - Bv,\ u - v\rangle \geq 0 \quad \forall\ u, v \in U$.*

Theorem 1 *Suppose that Assumptions 1-4 hold. Then the abstract semilinear state-dependent delay control system* (1), (2) *is $\mathfrak{T}$-controllable.*

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Method for the Approximate Solution of Constrained Optimal Control Problems in the Class of Piecewise-Constant Controls*

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A method for the approximate solution of constrained optimal control problems based on their approximation in the class of piecewise-constant control functions is considered. Necessary optimality conditions are formulated as a fixed-point problem for a special operator in the control space. An iterative method is proposed for finding an extremal control in the class of piecewise-constant controls. The efficiency of the method is illustrated by model examples.

Keywords: constrained optimal control problem, piecewise-constant control, control optimality condition, iterative method

Solving boundary value problems of the Pontryagin maximum principle and applying gradient methods in constrained optimal control problems [1] is associated with significant computational difficulties in selecting Lagrange multipliers and the need for control variation. In this paper, a fixed-point approach [2] is developed for an approximate solution in the class of constrained optimal control problems based on piecewise-constant control approximations.

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We consider a class of optimal control problems with a terminal constraint:

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x^0, \quad u(t) \in W \subset \mathbb{R}^m, \quad t \in T = [t_0; t_1], \tag{1}$$

$$J_0(u) = \varphi_0(x(t_1)) + \int_T F_0(x(t), u(t), t)dt \to \inf_{u \in V}, \tag{2}$$

$$J_1(u) = \varphi_1(x(t_1)) = 0, \tag{3}$$

where V is a set of piecewise-continuous functions with values in a convex compact set W. More complex problems with terminal, state, and mixed constraints can be reduced to the form (1)–(3) by standard methods of penalization for constraint violation.

Problem (1)–(3) is reduced to an auxiliary unconstrained Lagrange problem:

$$L^\lambda(u) = \lambda_0 J_0(u) + \lambda_1 J_1(u) \to \inf_{u \in V}$$

with a vector of Lagrange multipliers $\lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2, \lambda \neq 0$. In this paper, the non-degenerate case of the Lagrange problem $\lambda_0 = 1$ is considered.

To find an approximate solution in the class of piecewise-constant controls, a partition of the interval T into N subintervals $T_k = [t_{k-1}, t_k)$, $k = 1, 2, \dots, N$, such that $t_N = t_1$, is introduced. On each subinterval T_k, the control takes a constant value $u_k \in W$.

In the approximating problem, let us introduce the Pontryagin function:

$$H^\lambda(\psi, x, u, t) = \langle \psi, f(x, u, t) \rangle - F_0(x, u, t), \quad u \in W.$$

For any admissible control $u = \{u_1, \dots, u_N\}$, let $x(t, u)$ denote the solution to the state system (1), and $\psi^\lambda(t, u)$ denote the solution to the adjoint system:

$$\dot{\psi}(t) = -H_x^\lambda(\psi(t), x(t, u), u_k, t), \quad \psi(t_1) = -\varphi_x^\lambda(x(t_1, u)), \quad t \in T_k, \quad k = 1, 2, \dots, N,$$

where $\varphi^\lambda(x) = \varphi_0(x) + \lambda_1 \varphi_1(x)$.

Necessary optimality conditions for $u = \{u_1, \dots, u_N\}$ are represented in the form of a special fixed-point problem with a parameter $\alpha > 0$:

$$u_k = P_W\left(u_k + \alpha \int_{T_k} H_u^\lambda(\psi^\lambda(t, u), x(t, u), u_k, t)dt\right), \quad k = 1, 2, \dots, N,$$

$$\varphi_1(x(t_1, u)) = 0.$$

For the numerical solution of the fixed-point problem, an iterative process for $s \geq 0$ is considered:

$$u_k^{s+1} = P_W\left(u_k^s + \alpha \int_{T_k} H_u^\lambda(\psi^\lambda(t, u^s), x(t, u^s), u_k^s, t)dt\right), \quad k = 1, 2, \dots, N, \tag{4}$$

$$\varphi_1(x(t_1, u^{s+1})) = 0. \tag{5}$$

The proposed method for the approximate solution of the constrained optimal control problem is characterized by the satisfaction of all problem constraints at each iteration and does not require the calculation of the objective functional value or a control variation procedure.

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Biological Wave Solutions in the Bacillus Subtilis Chemotaxis Model*

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We consider a nonlinear parabolic system underlying the Bacillus subtilis chemotaxis model. The system describes the diffusion occurring between bacterial masses and a nutrient (chemoattractant). The model entails interesting problem statement when the density function (bacteria) has the form of a biological wave. We present some different solutions to the boundary value problem.

Keywords: degenerate parabolic system, boundary value problem, biological wave solution, chemotaxis

1 Nonlinear parabolic system

We consider the nonlinear parabolic system

$$u_t = u_{xx} - F(u,v), \quad v_t = \sigma(uvv_x)_x + G(v,u); \quad \sigma > 0, \tag{1}$$

which is the basis of the Bacillus subtilis chemotaxis model [1,2]. Chemotaxis is the chemically directed movement of bacteria. A bacteria population of density $v(t,x)$ is placed in a medium with a nutrient (chemoattractant) of concentration $u(t,x)$. Bacteria radically change their behavior and move toward the chemical stimulus (see fig. 1).

2 Boundary value problem

This situation entails boundary conditions

$$u(t,x)|_{x=a(t)} = u_0(t) > 0, \quad v(t,x)|_{x=a(t)} = 0. \tag{2}$$

System (1) has a degeneracy of the parabolic type at all points of the sufficiently smooth curve $x = a(t)$ (boundary of bacteria habitat). Using different methods we construct some solutions to boundary problem (1), (2). In each case, the density function $v(t,x)$ has the form of the biological wave.

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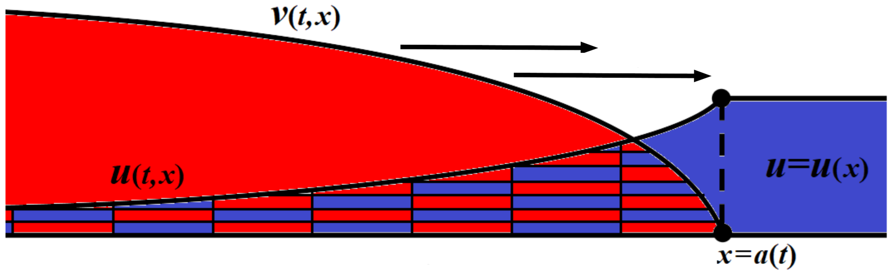

Figure 1: Bacteria $v \geq 0$ move toward the highest concentration $u > 0$. The function v has the form of a biological wave with the front $x = a(t)$. In the left half-neighborhood of the front, the v graph decreases, while the u graph, conversely, increases. To the right of the front, the concentration is stationary.

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Homogenization of a Layered Composite Reinforced with Thin Gradient-elastic Inclusions*

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Within the framework of the antiplane shear theory, the problem of equilibrium of an elastic composite body containing many thin gradient-elastic inclusions is considered. Based on a variational formulation of the problem, the behavior of a family of its solutions as ϵ tends to zero is investigated. The justification for the limit transition as $\epsilon \to 0$ is based on the application of the two-scale convergence method.

Keywords: linear elasticity, gradient theory of elasticity, composite material, thin inclusions, homogenization, two-scale convergence

1 The main results

Composite materials are widely used in civil engineering, mechanical engineering, medicine, and other industries. Therefore, the development of new mathematical models that most accurately describe the behavior of structures and components made from composite materials

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is of paramount importance. To develop effective models suitable for practical engineering applications, it is necessary to consider the heterogeneity of the material and the physical effects occurring within it at various spatial scales. This is particularly relevant for describing composites reinforced with thin fibers (threads, rods). We consider the static problem of antiplane shear of a layered elastic composite cross-linked with a finite number of rectilinear, parallel thin reinforcing gradient-elastic threads [1, § 5.4]. The original formulation contains a small positive parameter ϵ, characterizing the distance between adjacent threads. It is also assumed that the physical properties of the composite body depend on ϵ, which means they are contrasting and, consequently, highly heterogeneous. The well-posedness of the original formulation for fixed values of ϵ is guaranteed by known results [1, Theorem 5].

A homogenization procedure is performed on the original formulation, i.e., a limit transition in the equations as $\epsilon \to 0$. This transition is mathematically rigorously justified using the standard Allaire–Nguetseng two-scale convergence method [2] and a version of this method for convergence on thin periodic structures [3]. The result is the construction of a limiting averaged model of antiplane shear of a composite material. This model is mathematically correct. It is formulated on a macroscopic scale, i.e., at the characteristic scale of the entire composite body, where it is no longer necessary to consider the behavior of each individual thin inclusion, the physical characteristics of the thin inclusions are sublimated into the limiting effective coefficients and an effective average mechanical description of the entire composite body as a whole is given. A more detailed description of this study is published in [4].

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On the the Modified Method of Complex Autonomization and its Application to Beletsky Equation*

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The Beletsky differential equation describing the plane oscillations of a satellite in a Keplerian elliptical orbit is considered. To find periodic solutions to this equation, a modification of the complex autonomization method is used, which makes it possible to reduce the study of a non-autonomous s- order differential system to the study of an autonomous (s+2)-order differential system. It is shown that the proposed method for construction of periodic steady-state oscillations is successfully applied to a nonlinear differential equation and does not require small parameters in the nonlinear terms of the equation. Finding the coefficients of an approximate analytical solution reduces to a recurrent procedure.

Keywords: oscillations, periodic solutions, Beletsky equation

The differential equation of the following form is considered:

$$\dot{X} = F(X,t), \tag{1}$$

where $X \in \mathbb{R}^s$, $F(X,t)$ is a periodic function with respect to t. We will assume that $F(0,t) = 0$ and $F(X,t)$ is analytic on X with periodic coefficients.

By introducing new auxiliary variables:

$$y_1 = e^{it}, \quad y_2 = e^{-it}, \quad \frac{d}{dt} = D = iy_1\frac{\partial}{\partial y_1} - iy_2\frac{\partial}{\partial y_2} \tag{2}$$

the system (1) can be reduced to the form:

$$DX = F(X, y_1, y_2) = \sum_{i_1,\dots,i_s,j,k=1}^{\infty} C_{i_1,\dots,i_s,j,k}\, x_1^{i_1} \dots x_s^{i_s} y_1^j y_2^k. \tag{3}$$

We have expanded the right-hand side of the equation into a Maclaurin series with respect to all arguments. We will look for a periodic solution in the form of harmonic decomposition:

$$X(t) = X(y_1, y_2) = \sum_{j,k\geq 1} A_{jk} y_1^j y_2^k. \tag{4}$$

The original version of the method [1], [2] requires the Fourier expansion of F on t and then uses (2). Our modification uses the fact that the function F can be expressed in terms of y_1, y_2 in many ways, which eliminates the need for a Fourier series expansion in many cases. Substituting a solution of this form, we have an equation of two infinite polynomials. By

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equating the coefficients with the same monomials, we obtain systems of equations that allow us to find all the coefficients in turn. It can be shown [1] that these equations are the recurrent linear equations. To demonstrate the method, we consider the Beletsky equation [3], which describes the plane oscillations of a satellite relative to the center of mass in a Keplerian orbit:

$$(1+e\cos\nu)\frac{d^2\varphi}{d\nu^2}+n^2\sin\varphi\cos\varphi=2e\sin\nu\left(1+\frac{d\varphi}{d\nu}\right), \tag{5}$$

where e is the eccentricity of the orbit, $n^2=3\dfrac{A-C}{B}$, A, B, C are the main central moments of inertia, ν is the angle of the true anomaly and φ is the pitch angle of the satellite deviation relative to the orbital coordinate system. Then we have the following equation:

$$\frac{d^2\varphi}{d\nu^2}=-\frac{n^2\sin\varphi\cos\varphi}{1+e\cos\nu}+\frac{2e\sin\nu}{1+e\cos\nu}\left(1+\frac{d\varphi}{d\nu}\right). \tag{6}$$

Then we exclude the variable ν in equation (6) by introducing the auxiliary variables y_1, y_2 in accordance with equation (2):

$$D^2\varphi(y_1,y_2)=-2\left[\frac{n^2\sin\varphi\cos\varphi}{2+e(y_1+y_2)}+i\frac{e(y_1-y_2)}{2+e(y_1+y_2)}\left(1+D\varphi(y_1,y_2)\right)\right]. \tag{7}$$

Then we write Taylor expansion for functions on the right side of equation (7):

$$\frac{1}{2+e(y_1+y_2)}=\frac{1}{2}\frac{1}{1+\frac{e}{2}(y_1+y_2)}=\frac{1}{2}\left(1-\frac{e}{2}(y_1+y_2)+\frac{e^2}{4}(y_1+y_2)^2-...\right). \tag{8}$$

For the solution $\varphi(\nu)$ we have the following:

$$\begin{aligned}\varphi(\nu)=&\left[\frac{2e}{n^2-1}+\frac{4e^3n^2}{3(n^2-1)^4}\right]\sin\nu+\frac{4e^3(8n^4-14n^2+9)}{3(n^2-9)(n^2-4)(n^2-1)^3}\sin 3\nu+\\&+\left[\frac{3e^2}{(n^2-1)(n^2-4)}+\frac{2e^4(-9+185n^2-121n^4+17n^6)}{(-9+n^2)(-4+n^2)^2(-1+n^2)^4}\right]\sin 2\nu\\&+\frac{2e^4(45-43n^2+37n^4)}{(-16+n^2)(-9+n^2)(-4+n^2)(-1+n^2)^3}\sin 4\nu+...\end{aligned}$$

The advantage of this method is that it can be used manually and that numerical methods are not required in the intermediate stages. At the same time, the solution obtained by this method has coefficients that depend on the system parameters in a rational manner, which facilitates analysis. By calculating more coefficients, we can obtain any accuracy. Numerical investigation has shown that the relative error at values of n, far enough from the resonances, in our approximation is about 0.01%.

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Energy Estimates for Solutions to the Generalized Boussinesq Equation with Variable Coefficients*

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We study a class of sixth-order strictly pseudohyperbolic operators with variable coefficients [1], which includes, in particular, the generalized Boussinesq operator [2]. Energy estimates for fourth-order pseudohyperbolic operators with variable coefficients were established in [3].

Keywords: the differential operators unsolvable with respect to the highest time derivative, energy estimates, the Generalized Boussinesq Equation

1 The main results

For certain classes of equations that are not solved for the highest-order derivative, a number of important results are known in the theory of boundary value problems (see, for example, the monographs [1]). For the class of pseudohyperbolic equations, the Cauchy problem is fairly well understood in the constant coefficient case (see, e.g., [2]), whereas in the variable coefficient case there is only one result in the literature concerning energy estimates [3].

In this work, we continue the study of properties of pseudohyperbolic operators with variable coefficients. We considers the sixth-order differential operators unsolvable with respect to the highest time derivative:

$$\mathcal{L}(x;D_t,D_x)=\mathcal{L}^1(D_t,D_x)+\mathcal{L}^2(x;D_t,D_x),\quad t\in\mathbb{R},\quad x\in\mathbb{R}^n, \tag{1}$$

where $\mathcal{L}^1(D_t,D_x)$ is a homogeneous strictly pseudohyperbolic operator with constant real coefficients of the following form

$$\mathcal{L}^1(D_t,D_x)=\sum_{|\beta|=4}a^0_\beta D^\beta_x D^2_t+\sum_{|\beta|=5}a^1_\beta D^\beta_x D_t+\sum_{|\beta|=6}a^2_\beta D^\beta_x, \tag{2}$$

here $L^1_0(D_x)=\sum\limits_{|\beta|=4}a^0_\beta D^\beta_x$ is an elliptic operator.

The differential operator $\mathcal{L}^2(x;D_t,D_x)$ in (1) with real-valued variable coefficients has the form

$$\mathcal{L}^2(x;D_t,D_x)=\Big(\sum_{|\beta|=4}a^0_\beta(x)D^\beta_x+(a(x)+a)I\Big)D^2_t+$$
$$+\sum_{|\beta|=5}a^1_\beta(x)D^\beta_x D_t+\sum_{|\beta|=6}a^2_\beta(x)D^\beta_x, \tag{3}$$

while $a(x),\ a^k_\beta(x)\in C^\infty_0(\mathbb{R}^n), k=0,1,2,\ a>0$ is a constant.

The operator $\mathcal{L}^2(x;D_t,D_x)$ can be regarded as a perturbation of the pseudohyperbolic operator $\mathcal{L}^1(D_t,D_x)$.

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The following theorem holds:

Theorem 1. *There exists* $\gamma_0 > 0$ *such that if the coefficients* $a^k_\beta(x)$ *and* $a(x)$ *of operator (1), together with their derivatives up to and including the fifth order, are sufficiently small, then for any function* $u(t,x) \in W^{2,6}_{2,\gamma}(\mathbb{R}^{n+1})$, $\gamma > \gamma_0$, *such that there exists*

$$D_t^2 D_x^\beta u(t,x) \in L_{2,\gamma}(\mathbb{R}^{n+1}), |\beta| = 4, \tag{4}$$

the estimate holds

$$\begin{aligned} &\gamma \|(|\xi|^4 + a)(|\eta| + \gamma + |\xi|)\hat{u}_\gamma(\eta,\xi), L_2(\mathbb{R}^{n+1})\| \\ &\leq c\|\mathcal{L}(x; D_t, D_x)u(t,x), L_{2,\gamma}(\mathbb{R}^{n+1})\| \end{aligned} \tag{5}$$

with a constant $c > 0$ *independent of* $u(t,x)$.

Estimates (5) are analogues of energy inequalities for strictly hyperbolic operators (1).

We note that energy estimates of the form (5) can be used to study the well-posedness of the Cauchy problem for strictly pseudohyperbolic equations with variable coefficients

$$\begin{aligned} &\mathcal{L}(x; D_t, D_x) = f(t,x), \quad t > 0, \quad x \in \mathbb{R}^n, \\ &u|_{t=0} = \varphi_1(x), \quad D_t u|_{t=0} = \varphi_2(x), \end{aligned} \tag{6}$$

in the weighted Sobolev space $W^{2,6}_{2,\gamma}(\mathbb{R}^{n+1})$, $\gamma > 0$. In particular, Theorem 1 implies a uniqueness theorem for the solution of problem (5).

Theorem 2. *Let the assumptions of Theorem 1 hold. Then the Cauchy problem (6) cannot have more than one solution* $u(t,x) \in W^{2,6}_{2,\gamma}(\mathbb{R}^{n+1})$, $\gamma > \gamma_0$, *satisfying (4).*

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Robust Stability of a Parabolic System with Delay Based on a Lyapunov–Krasovskii Functional with Prescribed Derivative*

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In this work, we consider a parabolic equation with time delay under bounded perturbations. A Lyapunov–Krasovskii functional constructed with a prescribed derivative is used to study its stability. By introducing a modified functional, we obtain sufficient conditions for robust exponential stability. These conditions also provide explicit bounds on the admissible perturbations and exponential estimates of the solutions.

Keywords: parabolic equation, Lyapunov functionals, delay, robust stability

The Lyapunov–Krasovskii functional method is one of the main tools for the stability analysis of time-delay systems [1]. Within this framework, an important role is played by the construction of functionals with a prescribed derivative [1–2]. In this approach, the derivative of the functional along the system solutions is first prescribed as a non-positive quadratic form, and the functional itself is then reconstructed from this derivative. This method makes it possible to obtain explicit representations of Lyapunov functionals and to use them effectively for stability analysis.

We consider a boundary value problem for a parabolic equation with delay

$$\begin{cases} u_t(x,t) = au_{xx}(x,t) + (b+\delta(x,t))u(x,t-h), \quad x \in (0,1),\ t>0, \\ u(0,t) = u(1,t) = 0, \quad t \geq 0, \\ u(x,\theta) = \varphi(x,\theta), \quad \theta \in [-h,0],\ x \in [0,1]. \end{cases} \tag{1}$$

Here $a > 0$ and $b \in \mathbb{R}$ are system parameters, $h > 0$ is the delay, φ is the initial function, and $\delta(x,t)$ is a continuous perturbation satisfying $|\delta(x,t)| \leq \bar{\delta},\ x \in [0,1],\ t \geq 0$.

Solutions of the system are considered as functions with values in the space $L_2(0,1)$ with norm

$$\|u(\cdot,t)\|^2 = \int_0^1 u^2(x,t)\,dx.$$

For each $t \geq 0$, we define the state of the system by $\hat{u}_t(\theta) = u(\cdot, t+\theta),\ \theta \in [-h,0]$.

The goal of this work is to obtain conditions under which the zero solution of system (1) is robustly exponentially stable for all admissible perturbations δ.

Consider the quadratic functional

$$\omega(\varphi) = W_0\|\varphi(\cdot,0)\|^2 + W_1\|\varphi(\cdot,-h)\|^2 + W_2\int_{-h}^0 \|\varphi(\cdot,\theta)\|^2 d\theta,$$

where W_0, W_1, W_2 are positive constants.

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Next, we introduce a functional v constructed via a prescribed derivative. Its derivative along the solutions of the perturbed system has the form

$$\frac{d}{dt}v(u_t)\Big|_{(1)} = -\omega(u_t) + \Delta R_1(t) + \Delta R_2(t),$$

where the terms ΔR_1 and ΔR_2 are associated with the perturbation δ.

Thus, the presence of perturbations leads to additional terms in the derivative of the functional, which require further estimation. To obtain such estimates, we introduce the modified functional

$$v_\varepsilon(u_t) = v(u_t) + \varepsilon_1\|u(\cdot,t)\|^2 + \varepsilon_2\int_{t-h}^{t}\|u(\cdot,s)\|^2 ds, \quad \varepsilon_1, \varepsilon_2 > 0.$$

Estimate of the derivative and stability

The derivative of the functional v_ε along the solutions of the perturbed system admits the estimate

$$\frac{d}{dt}v_\varepsilon(u_t) \le -A\|u(\cdot,t)\|^2 - B\|u(\cdot,t-h)\|^2 - C(t)\int_{t-h}^{t}\|u(\cdot,s)\|^2 ds,$$

where the coefficients A, B, and $C(t)$ which depend on the system parameters, the functional, and the magnitude of the perturbation.

Theorem. *Suppose that the coefficients A, B, and $C(t)$, determined by the system parameters and the chosen functional, satisfy the conditions $A > 0$, $B > 0$, $C(t) > 0$, $t \ge 0$. Then the zero solution of the considered system is exponentially stable. Moreover, the following estimate holds for the solutions:*

$$\|u(\cdot,t)\|^2 \le \frac{\alpha_2}{\alpha_1}e^{-\sigma_{\mathrm{rob}}t}\left(\|\varphi(\cdot,0)\|^2 + \int_{-h}^{0}\|\varphi(\cdot,\theta)\|^2 d\theta\right), \tag{2}$$

where the coefficients α_1, α_2, and σ_{rob} are determined by the parameters of the functional and its derivative.

From the positivity conditions for the coefficients A, B, and $C(t)$, one obtains restrictions on the admissible value of the perturbation.

Lemma. *There exists $\delta^* > 0$ such that if $|\delta(x,t)| \le \bar{\delta} < \delta^*$, then*

$$A > 0, \quad B > 0, \quad C(t) > 0, \quad t \ge 0.$$

Consequently, under these constraints on the perturbation magnitude, the zero solution of the system is exponentially stable.

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Optimal Switching Control of Linear Systems via Mixed-Integer Linear Programming

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We consider the problem of steering a linear dynamical system to a prescribed target state in fixed time, where the control is restricted to a finite set of modes and transitions between modes incur a cost. We formulate this as a mixed-integer linear program (MILP) by exploiting the fact that, for linear systems, the terminal state is an affine function of the binary control sequence. The approach naturally handles practically relevant switching penalties, including the number of switches and travel-distance costs for spatially distributed actuators. We demonstrate the method on a rod-heating problem where a movable point-source heater must achieve a uniform temperature profile.

Keywords: optimal switching control, mixed-integer linear programming, bang-bang control, heat equation, sparse control

1 Problem Statement and MILP Reformulation

Consider a linear system with a discrete switching input:

$$\dot{x} = Ax + B_{y(t)}, \qquad x(0) = x_0, \qquad y(t) \in \{1, \dots, K\}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $B_1, \dots, B_K \in \mathbb{R}^n$. The objective is

$$J = \lambda_1 \|x(T) - x^*\|_1 + \sum_{\tau \in T(y)} g(y(\tau^-), y(\tau)) \ \to \ \min, \tag{2}$$

x^* is the target state, $T(y)$ is the set of switching moments of $y(\cdot)$, and $g(k, k')$ is the transition cost between modes k and k'. Two practically relevant choices are the switching penalty $g(k, k') = \lambda_2 \, \mathbf{1}_{k \neq k'}$ and the travel-distance penalty $g(k, k') = \lambda_2 |k - k'|$.

We split $[0, T]$ into M intervals of size $\Delta t = T/M$, assuming piecewise-constant control. The exact discrete dynamics are

$$x_{j+1} = A_d x_j + B_d^{y_j}, \qquad A_d = e^{A\Delta t}, \qquad B_d^k = \int_0^{\Delta t} e^{As} B_k \, ds. \tag{3}$$

Next we introduce binary variables $\alpha_{jk} \in \{0, 1\}$ with $\alpha_{jk} = 1 \iff y_j = k$, so that $B_d^{y_j} = \sum_{k=1}^K B_d^k \, \alpha_{jk}$. Unrolling (3) over all M steps yields

$$x_M = A_d^M x_0 + \sum_{j=0}^{M-1} \sum_{k=1}^{K} c_{jk} \, \alpha_{jk}, \qquad c_{jk} = A_d^{M-1-j} B_d^k \in \mathbb{R}^n. \tag{4}$$

Stacking all vectors c_{jk} as columns gives $C \in \mathbb{R}^{n \times MK}$, so (4) compacts to

$$x_M = A_d^M x_0 + C\alpha. \tag{5}$$

Thus the terminal state depends *affinely* on the binary switching sequence $\alpha \in \{0,1\}^{MK}$.

Below, for clarity, we concentrate only on the travel-distance penalty case (the switching penalty is treated analogously).

To handle norm term in the cost, we introduce nonnegative slack variables $e^+, e^- \geq 0$ with $e^+ - e^- = x_M - x^*$, so that $\|x_M - x^*\|_1 = \mathbf{1}^\top(e^+ + e^-)$. For the travel-distance penalty, introduce auxiliary variables $d_j \geq 0$ encoding $d_j \geq \pm|y_{j+1} - y_j|$. Since exactly one mode is active at each step, $\sum_k k\,\alpha_{jk} = y_j$, so $|y_{j+1} - y_j| = |\sum_k k(\alpha_{j+1,k} - \alpha_{jk})|$. The MILP becomes

$$\min_{\alpha,\, d,\, e^\pm} \quad \lambda_1 \mathbf{1}^\top(e^+ + e^-) + \lambda_2 \sum_{j=0}^{M-2} d_j$$

$$e^+ - e^- = x_0 + C\alpha - x^*,$$

$$\sum_k \alpha_{jk} = 1 \quad \forall j,$$

$$d_j \geq \sum_k k\,(\alpha_{j+1,k} - \alpha_{jk}), \quad d_j \geq -\sum_k k\,(\alpha_{j+1,k} - \alpha_{jk}) \quad \forall j,$$

$$\alpha_{jk} \in \{0,1\}, \quad d_j,\, e^\pm \geq 0 \quad \forall j, k.$$

The formulation contains $O(MK)$ binary variables and is efficiently solvable for moderate discretizations using modern MILP solvers such as HiGHS [1].

2 Example: Heating a Rod

We consider a rod of $n = 6$ nodes governed by a finite-difference discretization of the heat equation with Neumann boundary conditions:

$$\dot{x} = \frac{\kappa}{\Delta\xi^2} L_n x + P e_{y(t)}, \tag{6}$$

where L_n is the discrete Laplacian and $y(t) \in \{1, \ldots, 6\}$ denotes the heater position. Starting from $x_0 = 0$, the goal is the uniform profile $x^* = 10 \cdot \mathbf{1}$. Parameters: $\kappa = 2$, $\Delta\xi = 1$, $P = 7.5$, $T = 8$, $M = 40$. The energy balance $P \cdot T = n \cdot 10 = 60$ ensures feasibility. The transition cost is $g(k, k') = \lambda_2 |k - k'|$.

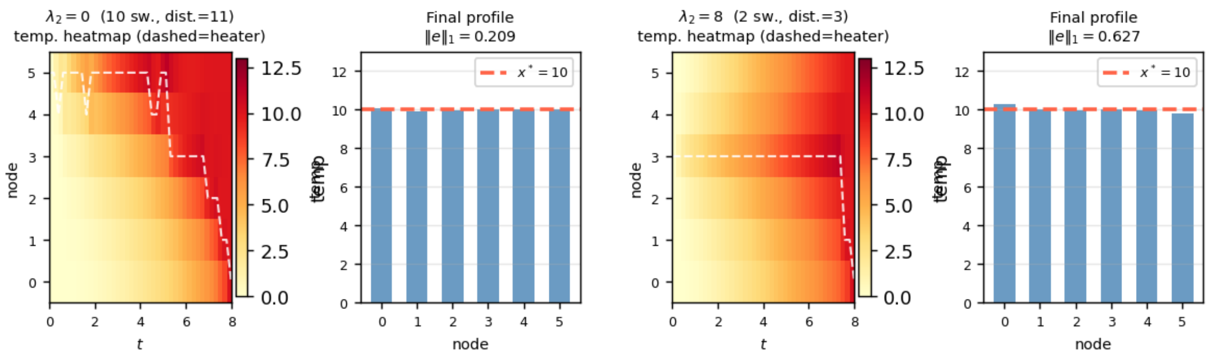

Figure 1: Rod heating for $\lambda_2 = 0$ (free movement, left pair) and $\lambda_2 = 8$ (penalized, right pair). Each pair: temperature heatmap with heater trajectory (dashed white) and final temperature profile vs. target $x^* = 10$.

For $\lambda_2 = 0$ the heater moves freely (10 switches, total distance 11) and achieves near-perfect uniformity ($\|e\|_1 = 0.21$). At $\lambda_2 = 8$ only 2 switches are made (distance 3), but the final profile is less uniform ($\|e\|_1 = 0.63$): the heater dwells in one place and relies on diffusion to spread heat to distant nodes.

Conclusions

For linear systems with discrete switching inputs, the terminal state depends affinely on the binary control sequence, allowing the optimal switching problem to be reformulated as a MILP. The framework naturally incorporates switching and actuator-motion penalties and applies directly to finite-dimensional discretizations of PDE control problems.

The use of MILP formulations in switching and hybrid optimal control is classical; see Bemporad–Morari [2] and Sager [3]. The present formulation emphasizes the explicit affine terminal-state representation and compact actuator-motion penalization for discretized distributed systems.

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Spatially Nonlocal Boundary Value Problems for Higher-order Parabolic Equations

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This paper investigates the solvability of some spatially nonlocal boundary value problems for fourth-order differential equations. In these problems, the boundary conditions represent a combination of the nonlocal boundary condition of A.A. Samarskii with variable coefficients. For the problems under study, existence and uniqueness theorems for regular solutions are proved – solutions possessing all generalized derivatives (in the sense of S.L. Sobolev) appearing in the corresponding equation.

Keywords: fourth-order differential equations, nonlocal problem, regular solution, existence, uniqueness

In the present work, we study the solvability of nonlocal problems with a general A.A. Samarskii condition for fourth-order parabolic equations of the form

$$u_t + u_{xxxx} + c(x,t)u = f(x,t)$$

Nonlocal boundary value problems with a generalized Samarskii–Ionkin condition for parabolic equations are analyzed in detail in the monograph by A.I. Kozhanov [1]. The closest in subject matter to the present work is the article [2], which is devoted to the study of nonlocal problems for a second-order parabolic equation with a Samarskii boundary condition.

We prove existence and uniqueness theorems for regular solutions possessing all necessary generalized derivatives in the sense of S.L. Sobolev. The obtained results provide new approaches to solving nonlocal problems in this area, which may be useful for applications in theoretical and applied research.

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Algorithms for Constructing an Optimal Strategy for the First Player on a Segment and a Polygon in a Linear Game Problem with Forbidden Situations

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An algorithm for constructing an optimal strategy on a segment and a polygon in a linear game problem with forbidden situations is proposed.

Keywords: game problem, players, support, algorithm

1 The main results

Many applied problems are associated with solving one-dimensional global optimization problems [1]. One-dimensional global optimization problems have been considered both in textbooks [2] and in scientific articles [1,3]. The results of these works cannot be applied to problems in which the objective function is not explicitly defined over the entire domain or is determined as the solution of other optimization problems. This class of problems includes game problems with linked variables (game problems with favorable situations, game problems with arbitrary situations, and game problems with forbidden situations), in which the first player chooses a strategy from a certain segment. Sometimes, when solving a specific optimization problem in a particular case (for example, when the dimensional of the space in which the problem is considered is small, e.g., 1, 2, 3, etc.), the use of a general known method/algorithm may be less efficient compared to a method/algorithm specifically developed for that particular case. Also, in studying n-dimensional optimization problems, one first considers them in special cases, i.e., for n=1,2, and then, whenever possible, these results are generalized to the remaining cases. Based on this, in the present paper it propos an algorithm for solving a game problem with connected variables, where the strategy of the first player is chosen from an interval as well as from a sabset of R^2. If the strategy of the first player is chosen from an interval, solving the considered problem requires solving no more than three linear programming problems. If the strategy of the first player is chosen from a polygon, the solution of the problem is determined by solving a finite number of linear programming problems using a graphical method.

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Solvability of Boundary Value Problems for Multidimensional Ultraparabolic Equations of the Fourth Order

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Keywords: ultraparabolic equations, integral analogues of the second boundary value problem, regular solutions, existence, uniqueness

Let us define the cylindrical domain Q as the set of points (x, y, t) such that $x \in \Omega = (0, 1)$, $y = (y_1, \dots, y_m) \in G \subset \mathbb{R}^m_y$, $t \in (0, T)$, $0 < T < +\infty$, and assume that G is a bounded domain with a smooth (in fact, infinitely differentiable) boundary Γ. Denote by Δ the Laplace operator in the variables $y_1, \dots, y_m$, defined by the equality

$$u_t + a(t)u_x + \Delta_y^2 u + c(x, y, t)u = f(x, y, t)$$

(where $c(x, y, t)$, $f(x, y, t)$, $a(t)$ are given functions). For this equation, we study the solvability of nonlocal problems with an integral-type condition in the spatial variable y. We prove existence and uniqueness theorems for regular solutions — solutions possessing all generalized (in the sense of S.L. Sobolev) derivatives that appear in the corresponding equation.

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Estimates of Solutions to a Class of Systems of Differential Equations of Neutral Type with Several Delays

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We consider neutral type systems with several delays. The aim of the work is to investigate the exponential stability of the zero solution, obtain estimates for attraction sets and establish Krein-type estimates for solutions to the initial value problem for these systems.

Keywords: time-delay systems, exponential stability, Lyapunov-Krasovskii functional

Consider nonlinear time-delay systems of neutral type of the following form

$$\frac{d}{dt}y(t) = Ay(t) + \sum_{j=1}^{m} \left(B_j y(t-\tau_j) + C_j \frac{d}{dt} y(t-\tau_j) + \int\limits_{t-\tau_j}^{t} D_j(t-s)y(s)ds \right)$$

$$+F(t, y(t), y(t-\tau_1), \dots, y(t-\tau_m)), \qquad t > 0, \tag{1}$$

where $\tau_j > 0$ are delays, A, B_j, C_j are $n \times n$ matrices with constant entries, $D_j(s)$ are $n \times n$ matrices with continuous entries on $[0, \tau_j]$, $j = \overline{1, m}$, $F(t, u_0, u_1, \dots, u_m)$ is a continuous vector-function satisfying Lipschitz condition with respect to u_0.

Using a special Lyapunov-Krasovskii functional introduced in [1], we establish conditions for the exponential stability of the zero solution, obtain estimates characterizing stabilization rates of solutions to (1) at infinity and estimates for attraction sets.

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Numerical Modelling of the Optimal Control Problem for a Degenerate Stochastic Nonlinear Filtration Equation*

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This paper considers a nonlinear control problem for stochastic Oskolkov equations of nonlinear filtering with a random initial input. The control function is assumed to be deterministic. The nonlinearity of the model complicates the search for an analytical solution, thereby necessitating numerical methods. An algorithm for the numerical solution of the optimal control problem is proposed, based on the projection, decomposition, penalty, and Ritz methods. Numerical results obtained from a program implementing the developed computational method are presented.

Keywords: optimal control problem, nonlinear filtering model, decomposition method, projection method, phase space method

1 Algorithm for solving the optimal control problem

Let us consider the complete probability space $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ and the set of real numbers $\mathbb{R}$, endowed with the Borel σ-algebra. Let $\mathfrak{D} \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary. Let us study the following stochastic model of nonlinear filtering:

$$(\lambda - \Delta) \overset{o}{\chi}_t - \alpha\Delta\chi + |\chi|^{p-2}\chi = u, \ p \geq 2, \tag{1}$$

$$\chi(\omega, s, t) = 0, \ \omega \in \Omega, \ (s, t) \in \partial\mathfrak{D} \times \mathbb{R}_+, \tag{2}$$

with the Showalter–Sidorov initial condition

$$(\lambda - \Delta)(\chi(\omega, s, 0) - \chi_0(\omega, s)) = 0, \ \omega \in \Omega, \ s \in \mathfrak{D}. \tag{3}$$

The function $u(s, t)$ is a deterministic external influence that needs to be determined. For the problem (1)–(3), random perturbation occurs only at the initial time (at $t = 0$); therefore, let us seek a solution to the problem as the sum of deterministic and stochastic components:

$$\chi(\omega, s, t) = x(s, t) + \eta(\omega, s, t).$$

Let us consider the special case where the model parameter is $p = 2k$. Then the initial problem (1)–(3) reduces to two problems: a deterministic problem and a stochastic problem. In this case, x is the solution to the deterministic problem:

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$$(\lambda-\Delta)x_t-\beta\Delta x+x^{2k-1}=u;\quad x(s,t)=0,\ (s,t)\in\partial\mathfrak{D}\times\mathbb{R}_+, \tag{4}$$

with the Showalter–Sidorov initial condition

$$(\lambda-\Delta)(x(s,0)-x_0(s))=0, \tag{5}$$

and η is the solution to the problem

$$(\lambda-\Delta)\overset{o}{\eta}-\beta\Delta\eta+\sum_{j=1}^{2k-1}C_j^{2k-1}x^{2k-1-j}\eta^j=0;\ \ \eta(\omega,s,t)=0,\ \omega\in\Omega,(s,t)\in\partial\mathfrak{D}\times\mathbb{R}_+, \tag{6}$$

with the Showalter–Sidorov initial condition

$$(\lambda-\Delta)(\eta(\omega,s,0)-\eta_0(\omega,s))=0. \tag{7}$$

For the problem (1)–(3), we consider the optimal control for solutions of (4), (5):

$$J(x,u)=\alpha\int_0^T\|x-z_d\|^{2k}_{L_{2k}(\mathfrak{D})}dt+(1-\alpha)\int_0^T\|u\|^q_{L_{\frac{2k}{2k-1}}(\mathfrak{D})}dt\to\inf,\alpha\in(0,1), \tag{8}$$

Let us develop an algorithm for solving problems (1)–(3), (8) which can be divided into seven stages:
1) Reduction of the problem into deterministic and stochastic components.
2) Based on the decomposition method, let us solve the equivalent problem by introducing an additional vector function $v(s,t)$:

$$(\lambda-\Delta)x_t-\beta\Delta x+v^{2k-1}=u,\quad x(s,t)=v(s,t).$$

3) Based on the projection method, let us represent the functions (x,v,u) as:

$$x_N(s,t)=\sum_{k=1}^N x_k(t)\varphi_k(s),\quad v_N(s,t)=\sum_{k=1}^N v_k(t)\varphi_k(s),\quad u_N(s,t)=\sum_{k=1}^N u_k(t)\varphi_k(s),$$

where $\{\varphi_k(s)\}$ are eigenfunctions of the Dirichlet problem for the operator $(-\Delta)$.
4) Let us form the functional J

$$J_{\theta,\varepsilon}(x,u,v)=\theta\cdot\alpha\int_0^T\|x-z_d\|^{2k}_{L_{2k}(\mathfrak{D})}dt+(1-\theta)\cdot\alpha\int_0^T\|v-z_d\|^{2k}_{L_{2k}(\mathfrak{D})}dt+$$

$$+(1-\alpha)\int_0^T\|u\|^{\frac{2k}{2k-1}}_{L_{\frac{2k}{2k-1}}(\mathfrak{D})}dt+r_\varepsilon\int_0^T\|x-v\|^{2k}_{L_{2k}(\mathfrak{D})}dt,\quad\theta\in(0,1),\quad\varepsilon\to0+,\quad r_\varepsilon\to+\infty$$

and find the functions (x,v,u) using the penalty method.
5) Let us substitute the function $x(s,t)$ into equation (6). Let $K=\{\mu_k\}$ be a monotonically decreasing numerical sequence such that $\sum\limits_{k=1}^{\infty}\mu_k^2<+\infty$. Let us form $\eta_0^l=\sum\limits_{k=1}^{N}\mu_k\eta_{0k}\varphi_k(s)$, and solve the problem (6), (7) with the generated η_{0k} to find the function $\eta(\omega,s,t)$.
6) Let us repeat the previous step ten times.

7) Let us conduct a statistical evaluation of the obtained results using the 3σ rule.
Let us implement the developed algorithm in the Maple 2017 software environment.

Example 1. Let us find a solution to the optimal control problem (1) – (3), (8) with parameters $p = 4, \beta = 1, \lambda = -1, \alpha = \frac{9}{10}, \theta = \frac{5}{6}, E(\eta_0) = 0, \sigma(\eta_0) = 2$ and the domain $\mathfrak{D} = (0, \pi)$. The initial state of the system is given by:

$$\chi_0 = x_0 + \eta_0, \quad x_0 = \sqrt{\frac{2}{\pi}} \cdot (2\sin(s) + 3\sin(2s)).$$

The desired state of the system is given by:

$$z_d = \sqrt{\frac{2}{\pi}} \cdot ((t^2 + 2t + 1)\sin(s) + (t^2 - t + 1)\sin(2s)).$$

As a result of the program execution, the functions (x, u) and the value of the functional J were obtained. The difference between the functions $x^N(s,t)$ and $z_d(s,t)$ is equal to

$$\Delta = \int_0^1 dt \int_0^\pi |\, x^N(s,t) - z_d(s,t)|^4 ds = 1.5342024.$$

Let us conduct ten experiments and find the trajectories of the solution (u, χ_l) (Fig. 1) of problem (1)–(3), (8). For statistical evaluation, let us construct a confidence interval (Fig. 2).

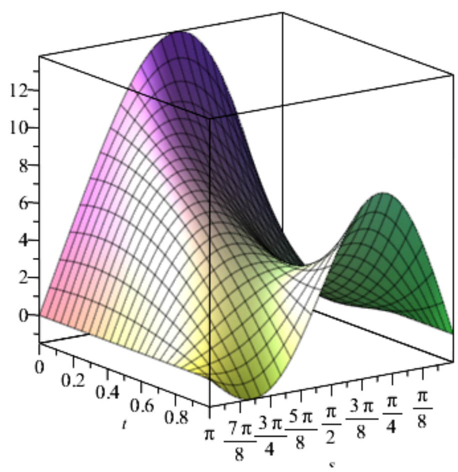

Figure 1: Control function $u^N(s,t)$

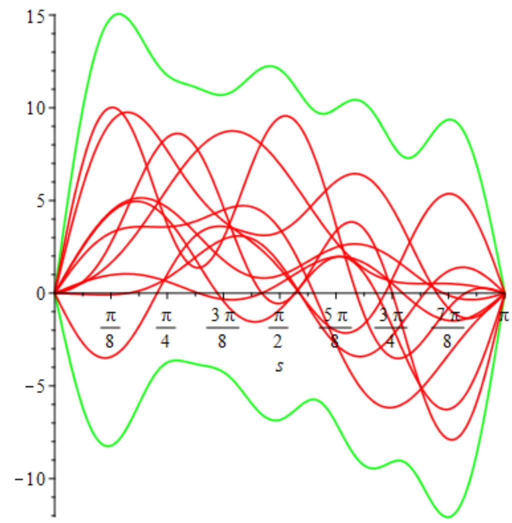

Figure 2: Trajectories of the solution of the stochastic process $\chi_l(\omega, s, t), t = 0.1$

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Investigation of the Structure of the Phase Manifold of One Sobolev-type Equation*

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This paper investigates the structure of the phase manifold of the I-beam deformation equation with a Dirichlet boundary condition, which lies on a smooth Banach manifold with singularities (the Whitney k-assembly) depending on the problem parameters. The mathematical model is studied in the case where the dimension of the operator kernel at the time of the time derivative is arbitrary.

Keywords: Sobolev type equations, Whitney assemblies, phase space method

1 Research of phase manifold features

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a boundary of class C^∞. Consider the Hoff model

$$(\mu + \Delta)u_t = \alpha u + \beta u^3, \; x \in \Omega, t \in (0, T), \tag{1}$$

$$u(x, t) = 0, \; (x, t) \in \partial\Omega \times (0, T). \tag{2}$$

The Hoff model describes the dynamics of deformation of an I-beam (in the case $n = 1$). The unknown function $u = u(x, t), \; x \in \Omega, \; t \in (0, T)$, describes the deviations of the beam from the equilibrium position. The parameter $\mu \in \mathbb{R}$ characterizes the longitudinal load on the beam, and the parameters $\alpha, \beta \in \mathbb{R}$ characterize the properties of the beam material. In the work [1], a study of the structure of the phase manifold of the equation (1) with the condition (2) was already carried out and the conditions imposed on the parameters α, β in case $\dim\ker(\mu + \Delta) = 2$ were identified. The purpose of this study is to generalize the results of studies on the structure of the phase manifold in the case $\dim\ker(\mu + \Delta) = m, m \in \mathbb{N}$.

In the paper [2] it is shown that the phase manifold of equation (1) with condition (2) is a simple Banach C^∞-manifold in the case $\alpha\beta > 0$ and takes the form:

$$\mathfrak{B} = \left\{ u \in L_4(\Omega) : \int\limits_{\Omega} (\alpha + \beta u^2) u \varphi_l dx = 0, \, l : \mu = \lambda_l \right\},$$

where $\{\lambda_l\}$, $\{\varphi_l\}$ are the eigenvalues and eigenfunctions of the homogeneous Dirichlet problem for operator $(-\Delta)$. It was also shown in paper [3] that, in the case of $\alpha\beta < 0$, the phase space equations (1) may contain a Whitney 2-assembly in the case where $\dim\ker(\mu + \Delta) = 1$.

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For example, for $\dim\ker(\mu+\Delta)=1$ the eigenvalue λ_l under consideration corresponds to the eigenfunction $\varphi_l(x,y)$, then u can be represented as $u=s\varphi_l+u^{\perp}$, $u^{\perp}\in\mathfrak{U}^{\perp}=\{u\in L_4(\Omega):\langle u,\varphi_l\rangle=0\}$. Then the set $\mathfrak{B}$ is C^{∞} diffeomorphic to the set [1]

$$\mathfrak{B}=\{(s,u^{\perp})\in\mathbb{R}\times L_4(\Omega): s^3\|\varphi_l\|^4_{L_4(\Omega)}+3s^2\int\limits_{\Omega}\varphi_l^3u^{\perp}\,dx+ \\ +\ s(3\int\limits_{\Omega}\varphi_l^2(u^{\perp})^2\,dx+\alpha\beta^{-1}+\int\limits_{\Omega}\varphi_l(u^{\perp})^3\,dx=0\}.$$

Let us formulate a theorem on the structure of the phase manifold of equation (1) with the condition (2).

Theorem 1. *(i) Let $\alpha\beta>0$, then the phase manifold of equation (1) with condition (2) is a simple Banach C^{∞}-manifold modeled by the subspace complementary to* $\ker(\mu+\Delta)$*;*
(ii) Let $\alpha\beta<0$, then the set $\mathfrak{B}$ forms a Whitney 2-assembly.

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On Properties of Solutions to Nonautonomous Time-delay Systems

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We consider some classes of nonautonomous time-delay systems Using specail Lyapunov–Krasovskii functionals, some estimates for solutions are established. The obtained estimates allow us to conclude whether the solutions are stable.

Keywords: time-delay systems, variable coefficients, estimates for solutions, stability, Lyapunov–Krasovskii functional

We consider some classes of systems of nonautonomous time-delay equations and study asymptotic properties of solutions to the systems. A lot of works have been devoted to the study of the problem on stability of solutions to time-delay equations. Researchers use various approaches; for example, D-subdivision method, method of Razumikhin functions, method of Lyapunov–Krasovskii functionals, comparison method, Azbelev W-method, test-method, etc. The authors of the most of the works consider autonomous equations. The study of asymptotic properties of solutions to nonautonomous equations is much more complicated.

This work continues our research of the stability of solutions to nonautonomous nonlinear delay equations (see, for example, [1–5]). Using special Lyapunov–Krasovskii functionals, for the mentioned class of time-delay systems, we establish conditions for the exponential stability of the zero solution, obtain estimates of exponential decrease of solutions at infinity and estimates for attraction sets.

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Some Extremal Problems for Minimizing Nondeterministic Finite Automata

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We consider a set of auxiliary algorithms used for the vertex minimization of nondeterministic finite automata of large dimensions. We present: (i) our interpretation of the algorithm for constructing a universal finite automaton that, when implemented naturally, runs faster than the algorithms known to us; (ii) consideration of the well-known Waterloo automaton based on this interpretation; (iii) a description of equivalence classes of ordinary languages based on a special algorithm.

Keywords: regular languages, nondeterministic finite automata, large dimension, minimization problem

In this paper, we consider the set of algorithms related to minimization of nondeterministic finite automata. The minimization algorithms proposed in the paper, as well as relevant theoretical questions, are of interest to all theoretical computer science. The main reason for the interest is as follows. Since we are considering a difficult problem [1], there are no acceptable brute force algorithms for large dimensions. As a result, it is highly desirable to develop heuristic algorithms, and heuristics used for one of these tasks can easily be applied to other difficult discrete optimization problems. For the specific algorithms, we first use the universal finite automaton for the given regular language [2], etc. Apparently, for the first time, the title "minimization using the universal automaton" was used in [3]. However, there are inaccuracies in that work, in particular, the so-called Kiel automaton proposed there does not have the necessary properties of the well-known Waterloo automaton; for details, see below.

In general, this paper can be considered a continuation of several works by the author; among these papers, we note [4]–[8]. However, we can say that this article is located between the theory of [4], [5] and the actual practical algorithms of [7], [8].

The following fact can also be mentioned as motivation. A very interesting example of automaton minimization (the Waterloo automaton) was published more than 50 years ago in [9], but since then, detailed studies of this example are unknown to us. We can also provide a motivation related to universal automata; note a phrase from [2]: "the origin of the universal automaton is not completely clear". We believe that this phrase is used because it is not only unclear who defined the first such automaton, but also because a clear construction algorithm was not described; moreover, even the definition given in [2] can hardly be considered constructive: it requires refinement of the corresponding algorithm. Then we present our interpretation of the algorithm for constructing a universal finite automaton, which, when implemented naturally, works faster than the algorithms we know (implemented in both C++ and GAP). We consider a simple example as well as the Waterloo automaton, and for them, we create the universal automata using our definition.

We also present the description of the equivalence classes of regular languages based on a special binary relation # defined for the given regular language. On this relation, we provided the description of the so-called complete automaton.

Now, let us consider the brief description of the results of our papers [4]–[8].

In this section, we use the concept of "semilattice" not for the semilattice of all canonical automata corresponding to the relation # under consideration, but in the usual sense, i.e. as a semilattice at the intersection of some subsets of the entire set of grids. There may be several similar semilattices for some language under consideration.

It is enough to simply describe the calculations associated with the Kiel automaton. Its #-relation matrix contains 13 grids, of which 800 covering sets of grids can be made, containing from 4 to 13 grids. Based on these covering sets, 4 different semilattices can be constructed, but all their elements (covering automata) will be *equivalent* to the original Kiel automaton.

We obtained 13 grids, of which 800 covering sets can be made, containing from 4 to 13 grids. These sets can be divided into 4 different semilattices, however, all their elements will be equivalent to the original Kiel automaton. That is why we claim that the Kiel automaton *does not have* the walibad property.

The characteristics of Waterloo automaton related to the semi-lattices of its covering automata are much more interesting. The main walibad property of the Waterloo automaton is the presence of 8 *sets of equivalent covering automata* that are not equivalent to the original automaton; 4 *semilattices* correspond to them. For it, 3 of the considered sets are included in the first semilattice, 2 sets are included in the second semilattice (partially in the third), one set is included only in the third semilattice, and 2 last sets are included in the fourth semilattice (partially also in the third).

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Compactness of Weighted Sobolev Trace Operators in Outward Cuspidal Domains*

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We study weighted Sobolev trace embeddings and nonlinear Steklov-type spectral problems in outward γ-cuspidal domains. The cusp singularity prevents the

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direct use of the classical compact trace theory in unweighted spaces. To compensate for this effect, we introduce a boundary weight determined by the geometry of the cusp and prove compactness of the corresponding weighted trace operator.

Keywords: Sobolev spaces, trace operators, cuspidal domains, composition operators, weighted Steklov eigenvalues

We study weighted Sobolev trace embeddings and nonlinear Steklov-type spectral problems in outward γ-cuspidal domains. Let

$$\Omega_\gamma = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_n < 1,\ 0 < x_i < x_n^\alpha,\ i = 1, \dots, n-1\},$$

$$\alpha = \frac{\gamma - 1}{n-1},$$

where $1 < p < n < \gamma < \infty$. The cusp singularity prevents the direct use of the classical compact trace theory in unweighted spaces. To compensate for this effect, we introduce a boundary weight determined by the geometry of the cusp and prove compactness of the corresponding weighted trace operator.

Our method is based on composition operators on Sobolev spaces. We flatten the cusp by explicit homeomorphisms from a Lipschitz model domain to Ω_γ and factor the trace operator through the standard trace embedding on the model domain. This gives a geometric explanation of the weight and leads to the compact embedding

$$T : W^{1,p}(\Omega_\gamma) \to L^q(\partial\Omega_\gamma, w_\gamma), \qquad 1 < q < \frac{p(n-1)}{n-p},$$

with

$$w_\gamma(x) = x_n^\beta,$$

$$\beta = \frac{(\gamma - n)(1 + p(n-2))}{(n-p)(n-1)}.$$

At the critical exponent $q = \frac{p(n-1)}{n-p}$ the same trace operator is continuous. We also derive an unweighted compact trace theorem in terms of an effective boundary dimension of the cusp.

The second result concerns sharpness. In the class of power weights x_n^θ, we prove a necessary condition for boundedness of the embedding into $L^q(\partial\Omega_\gamma, x_n^\theta)$. In particular, at the critical trace exponent one must have $\theta \geq \beta$, so the above weight is optimal in the power scale.

As an application, we consider a weighted Schrödinger–Steklov (p,q)-eigenvalue problem on Ω_γ.

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = 0 & \text{in } \Omega_\gamma, \\ |\nabla u|^{p-2}\nabla u \cdot \nu = \lambda\, w_\gamma\, \|u\|^{p-q}_{L^q(\partial\Omega_\gamma, w_\gamma)}\, |u|^{q-2}u & \text{on } \partial\Omega_\gamma. \end{cases}$$

The compactness of the weighted trace operator implies existence of the first nontrivial eigenpair, yields a variational characterization of the principal eigenvalue, and identifies it with the reciprocal p-th power of the optimal weighted trace constant. Thus the weighted trace theory developed here provides a natural analytic framework for nonlinear boundary spectral problems in singular domains.

Functional Volterra Integral Equations Solved via an Enhanced ADM–Aitken Framework with Stochastic Validation*

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This study presents a novel computational framework to solve functional Volterra integral equations by integrating the Adomian Decomposition Method (ADM) with Aitken's acceleration and stochastic arithmetic validation. The approach analyzes error estimation and convergence while maintaining numerical reliability via the CESTAC (Contrôle et Estimation Stochastique des Arrondis de Calculs) method and the CADNA (Control of Accuracy and Debugging for Numerical Applications) library. A benchmark example shows that the ADM–Aitken framework improves accuracy and reduces computation time by 25–50%, while CESTAC enables determination of the optimal number of iterations and minimal error.

Keywords: functional Volterra integral equations, Adomian Decomposition Method, Aitken Acceleration, CESTAC method, CADNA library

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Solving Nonconvex Quadratic Programming Problems Using a Branch and Bound Method with Inner and Outer Box Approximations*

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We investigate a nonconvex quadratic programming problem with the bounded feasible set. This problem has solutions only on the boundary of the feasible set. Using the separable form of the problem, we propose a branch and bound procedure, where only the difference between outer and inner box approximations of the feasible region is considered and partitioned further. The method is also adapted for finding all global solutions of the problem.

Keywords: quadratic programming, global optimization, branch and bound

Consider a nonconvex quadratic programming problem, i.e. the problem of minimizing a nonconvex quadratic function subject to linear constraints. The feasible set is assumed to be bounded. Since every quadratic function can be reduced to the separable form using a linear transformation [1], our problem has the form

$$\sum_{i=1}^{n} \lambda_i x_i^2 + \langle c, x \rangle \to \min, \qquad Ax \leqslant b, \tag{1}$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\lambda_i < 0$ for at least one $i = 1, \dots, n$.

Problem (1) has solutions only on the boundary of the feasible polyhedron. Using this fact, we define the minimal box-constrained set circumscribed around the feasible set and the maximal box-constrained set inscribed in the feasible set [2,3]. Then we partition the difference of these box approximations into several box-constrained subsets, as the interior of the inner approximation does not contain any solution. For each partition element, we use the separable form of the objective function to compute the lower bound [4]. Those subsets whose lower bound exceeds the best known objective value are to be discarded. This process is continued according to the classical branch and bound method until a global solution of (1) is found [5].

Since the branch and bound algorithm allows us to store all subsets that contain global minima, the proposed method is additionally adapted for finding all global solutions of problem (1).

The results of numerical testing on random problems and instances from the literature are presented.

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Solvability Conditions for One Pseudohyperbolic System*

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The Cauchy problem for one system unsolvable with respect to the highest time derivative is considered. The unique solvability of the Cauchy problem in Sobolev spaces is proved and estimates for the solution are obtained.

Keywords: pseudohyperbolic system, Cauchy problem, transverse flexural-torsional vibrations

In this paper we consider the Cauchy problem for a system of differential equations

$$\mathcal{L}(D_t, D_x)U = F(t,x), \quad t > 0, \, x \in R, \quad U\big|_{t=0} = D_t U\big|_{t=0} = 0, \tag{1}$$

where the operator $\mathcal{L}(D_t, D_x)$ has the form

$$\mathcal{L}(D_t, D_x) = \begin{pmatrix} I - \alpha_1 D_x^2 & 0 & \varepsilon_1 \\ 0 & I - \alpha_2 D_x^2 & -\varepsilon_2 \\ \varepsilon_1 & -\varepsilon_2 & I - \alpha_3 D_x^2 \end{pmatrix} D_t^2 + \begin{pmatrix} l_{11} \, l_{12} \, l_{13} \\ l_{12} \, l_{22} \, l_{23} \\ l_{13} \, l_{23} \, l_{33} \end{pmatrix} D_x^4,$$

$\varepsilon_1^2 + \varepsilon_2^2 = 1$, $\alpha_j > 0$, $j = 1, 2, 3$, and the matrix (l_{ij}) is positive definite.

System (1) is unsolvable with respect to the highest time derivative and belongs to the class of pseudohyperbolic systems [1]. For the first time, system (1) was considered in the monograph [2] for $l_{ij} = 0, i \neq j$. It arises when describing transverse flexural-torsional vibrations of an elastic rod. In the course of the study, the following theorem was proved.

Theorem. *Let $F(t,x) = (f_1(t,x), f_2(t,x), f_3(t,x))^T \in W_{2,\gamma}^{0,1}(R_+^2)$, $\gamma > 0$, be such that $x^2 F(t,x) \in L_{2,\gamma}(R_+; L_1(R))$ and for every $t > 0$ the following conditions hold:*

$$\int_R F(t,x)dx = \int_R xF(t,x)dx = 0, \quad j = 1, 2, 3. \tag{2}$$

Then the Cauchy problem (1) has a unique solution $U(t,x)$ in the space of vector-functions $W_{2,\gamma}^{2,4}(R_+^2)$ such that $D_t^2 D_x^2 U \in L_{2,\gamma}(R_+^2)$.

In the work [3] necessary and sufficient conditions for the solvability of problem (1) were obtained in the special case for $\alpha_1 = \alpha_2 = \alpha_3$ and $(l_{ij}) = lI$ with $l > 0$. We show that conditions of the form (2) are close to necessary conditions for solvability.

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Optimisation of Spatial Hashing Parameters for Neighbour Search in Particle Systems

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This paper considers the problem of selecting spatial hashing parameters for neighbour particle search. It is shown that the performance of the method is determined not only by the query radius, but also by the grid cell size, which affects both the cost of updating the data structure and the execution of queries.

Keywords: spatial hashing, neighbour search, particles, parameter optimisation, computational performance

1 The main results

In the simulation of systems with a large number of particles, neighbour search is one of the most computationally expensive operations. The use of spatial hashing reduces the computational cost in comparison with direct pairwise search; however, the efficiency of this approach depends strongly on the choice of grid cell size. Cells that are too small lead to increased update costs, since moving particles frequently transition between cells. Cells that are too large likewise degrade performance, as more particles must be checked during each query [1].

The experiments carried out show that the optimal cell size cannot be reduced to a simple geometric estimate and is determined not only by the query radius, but also by particle density and the nature of particle motion. Under the conditions considered, the optimal values lie in the region of $h \approx 40$, which indicates the need to tune the parameters with regard to system dynamics rather than the search region alone [2].

The dependence of the optimal cell size on the number of particles is shown in Fig. 1, where r denotes the radius of the square query. It can be seen that, as particle density decreases, the optimal cell size increases. At the same time, the effect itself is moderately pronounced, which suggests the existence of a fairly stable range of parameters that ensures high performance of the method.

Spatial hashing is an efficient method for neighbour particle search, and the choice of cell size has a significant effect on computational speed.

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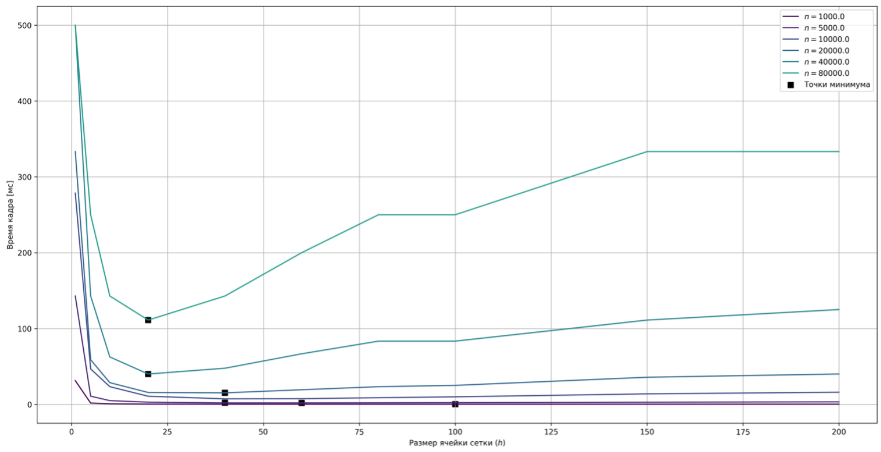

Figure 1: Optimal cell sizes h for $r = 20$ and different numbers of points n

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A Criterion for the Complete Continuity of the Urysohn Operator in Spaces of Continuous Banach-Valued Functions*

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Developing the authors' research on linear and nonlinear integral functionals and operators [1,2,3,4,5], we establish a criterion for the complete continuity of the nonlinear Urysohn integral operator in spaces of continuous Banach-valued functions, which generalizes the corresponding classical result for spaces of scalar functions [6, p. 372].

Keywords: Banach space, completely continuous operator, Urysohn operator

1 Basic notations

Let Ω be a closed, bounded, connected subset of $\mathbb{R}^n$ endowed with the classical Lebesgue measure μ.

Let Ω be a closed, bounded, connected set of $\mathbb{R}^n$ endowed with the Lebesgue measure.

For an arbitrary normed space Z, we use the following notation: 0_Z denotes the zero element of Z; $\|\cdot\|_Z$ denotes the norm in Z; $B_r[Z]$ denotes the closed ball in Z centered at 0_Z with radius r and Z^* denotes the dual Banach space of Z.

* The research is supported by SIDA under the subprogram *Capacity Building in Mathematics, Statistics and Its Applications* (Subprogram 1.4.2).

We assume that X and Y are real separable Banach spaces, with the additional condition that Y does not contain a copy of c_0. The value of a functional $g \in Y^*$ at a point $y \in Y$ is denoted by $\langle y, g\rangle$.

We denote by $C(X)$ the Banach space of continuous functions $u : \Omega \to X$, equipped with the norm $\|u\| = \sup_{t\in\Omega} \|u(t)\|_X$, and by $L_\infty(X)$ the Banach space of (classes of μ-equivalent) measurable essentially bounded functions $u : \Omega \to X$, with the norm $\|u\|_\infty = \operatorname*{ess\,sup}_{t\in\Omega} \|u(t)\|_X$.

The main object of our study is the nonlinear Urysohn integral operator K defined by

$$(Ku)(t) = \int_\Omega k(t, s, u(s))\, ds, \qquad t \in \Omega, \tag{1}$$

where $k : \Omega^2 \times X \to Y$, and the integral is understood in the Pettis sense [7, p. 54].

We assume that the kernel k satisfies the following conditions:

I. For any $t \in \Omega$, the function $k(t, \cdot, \cdot)$ satisfies the Carathéodory conditions: $k(t, \cdot, x)$ is measurable for each $x \in X$, and $k(t, s, \cdot)$ is continuous for almost every $s \in \Omega$;

II. $k(\cdot, \cdot, u_0(\cdot)) \equiv 0_Y$ for any $u_0 \in L_\infty(X)$;

III. $k(t_0, \cdot, \cdot) \equiv 0_Y$ for any $t_0 \in \Omega$.

Condition II is not too restrictive. Indeed, let K_1 be a Urysohn operator with kernel k_1 such that for some $u_0 \in L_\infty(X)$ (e.g., for $u_0 \equiv 0_X$) the condition $v_0 := Ku_0 \in C(Y)$ is easily verified. Then, the study of K_1 reduces to the study of the operator K with kernel $k(t, s, x) = k_1(t, s, x) - k_1(t, s, u_0(s))$, which satisfies Condition II, since $K_1u = Ku + v_0$.

Condition III is satisfied for an important class of Volterra nonlinear integral operators. For $\Omega = [a, b]$, the operator

$$(K_2u)(t) = \int_a^t k_2(t, s, u(s))\, ds, \qquad t \in [a, b],$$

is a special case of the operator (**??**), in which k satisfies Condition III with $t_0 = a$.

We define the quantity $\omega(t_1, t_2, r)$ for $t_1, t_2 \in \Omega$ and $r > 0$ by

$$\omega(t, s, r) = \sup_{g\in B_1[Y^*]} \int_\Omega \sup_{x\in B_r[X]} |\langle k(t_1, s, x) - k(t_2, s, x), g\rangle|\, ds.$$

An operator $A : U \to V$ (where U and V are normed spaces) is called *compact* if it maps bounded sets into precompact sets, and *completely continuous* if it is continuous and compact.

2 Main result

Theorem 1. *The following statements are equivalent:*

(A) *operator K acts from $C(X)$ to $C(Y)$ and is compact;*

(B) *operator K acts from $L_\infty(X)$ to $C(Y)$ and is completely continuous;*

(C) $\forall r > 0\ \ \forall t_0 \in \Omega \ : \ \lim_{t\to t_0} \omega(t, t_0, r) = 0$;

(D) $\forall r > 0 \ : \ \lim_{\delta\to 0^+} \sup_{t_1,t_2\in\Omega,\ \|t_1-t_2\|\le\delta} \omega(t_1, t_2, r) = 0.$

The research is supported by SIDA under the subprogram *Capacity Building in Mathematics, Statistics and Its Applications* (Subprogram 1.4.2).

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On the Global Solvability of the Wave Equation on a Geometric Graph with Second Derivative in Transmission Conditions

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We consider a mixed problem for the wave equation on a connected oriented geometric graph. The transmission conditions at the interior vertices involve the second time derivative of the unknown function and satisfy generalized nonlinear conditions of the Osgood type. Under certain regularity and growth assumptions on the nonlinearities, we prove the existence, uniqueness, and global extendability of classical solutions.

Keywords: wave equation, geometric graph, transmission conditions, global solvability, Osgood condition

1 The main results

Let $\Gamma \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a connected oriented geometric graph as defined in the monograph [1]. Denote by $\mathcal{J}(\Gamma)$ the set of interior vertices of Γ; then $\mathcal{R}(\Gamma) = \Gamma \ \mathcal{J}(\Gamma)$ is the union of its edges. We assume that all vertices of Γ are interior, i.e., $\partial\Gamma = \emptyset$.

We study the initial value problem for the wave equation

$$u_{xx}(x,t) = u_{tt}(x,t), \quad x \in \mathcal{R}(\Gamma),\, t \in (0,+\infty), \tag{1}$$

with initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \qquad x \in \Gamma, \tag{2}$$

general transmission conditions at the vertices $a \in \mathcal{J}(\Gamma)$

$$u_{tt}(a,t) = g_a\left(t, u(a,t), u_t(a,t), (u^+_{h(a,b)})_{a\leftrightarrow b}(a,t)\right), \quad t \in [0,+\infty), \tag{3}$$

and the regularity inclusion

$$u\big|_{\overline{\gamma}\times[0,+\infty)} \in C^2\left(\overline{\gamma} \times [0,+\infty)\right). \tag{4}$$

Here $u : \Gamma \times [0,+\infty) \longrightarrow \mathbb{R}$ is the unknown function; φ, ψ, g_a are given. The symbol $(u^+_{h(a,b)})_{a\leftrightarrow b}$ denotes the set of right partial derivatives with respect to the spatial variable along the direction h (from a to b) for all a adjacent to b. We consider classical solutions of (1)–(4) and assume that the consistency conditions hold.

Theorem. *Let $n = \deg(a) + 2$ and suppose that for each $a \in \mathcal{J}(\Gamma)$ and $t \in [0,+\infty)$ the functions g_a satisfy the following conditions:*

1) There exists a positive constant M_{g_a} such that

$$|g_a(t,0,0,\dots,0)| \le M_{g_a}.$$

2) There exists a continuous nondecreasing function $\omega_a : [0,+\infty) \longrightarrow [0,+\infty)$ with $\omega_a(0) = 0$ such that

$$|g_a(t,x_1,x_2,\dots,x_n) - g_a(t,y_1,y_2,\dots,y_n)| \le \eta_a(t)\cdot\omega_a\left(\sum_{k=1}^{n}|x_k - y_k|\right),$$

where $\eta_a(t) \ge 0$ is a continuous function, and the Osgood condition holds (see [2])

$$\int_{0^+} \frac{ds}{\omega_a(s)} = \infty.$$

3) There exists a continuous nondecreasing function $\kappa_a : [0,+\infty) \longrightarrow [0,+\infty)$ with $\kappa_a(0) = 0$ such that

$$|g_a(t,x_1,x_2,\dots,x_n)| \le \phi_a(t)\cdot\kappa_a\left(\frac{|x_1|}{t+1}\right)\cdot x_2,$$

where $\phi_a(t) \ge 0$ is a continuous function, and

$$\int^{+\infty} \frac{ds}{s\cdot\kappa_a(s)} = \infty.$$

Then the solution of problem (1)–(4) exists, is unique, twice continuously differentiable on $[0,+\infty)$, and can be extended arbitrarily far.

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On Optimization for Obtaining Two-Qubit Gates under Unitary Dynamics with Controls Formed with Chebyshev Polynomials

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The talk is about an approach being developed by the author in this direction.

Keywords: quantum optimal control, Chebyshev polynomials, quantum gates

Dedicated to the 205th anniversary of the birth of Acad. P.L. Chebyshev

1. Introduction. Quantum optimal control and, in particular, synthesis of quantum gates are important actual scientific directions. This talk describes some approaches being developed by the author and devoted to some topics for synthesis of quantum gates under non-dissipative dynamics. Consider some m-qubit case with $U \in SU(N)$, $N = 2^m$, and $W \in SU(N)$ or $W = \mathrm{proj}_{SU(N)}(V)$, $V \in U(N)$. We can consider the following usual differentiable infidelities: $E_{\mathrm{abs}}(U;W) = 1 - \frac{1}{N^2}\left|\mathrm{Tr}(W^\dagger U)\right|^2 \in [0,1]$ (used, e.g., in [1]), $E_{\mathrm{Re}}(U;W) = 1 - \frac{1}{N}\mathrm{Re}[\mathrm{Tr}(W^\dagger U)] \in [0,1]$ (used, e.g., in [2]). For a comparing analysis, one can take the bi-invariant metric $E_{\mathrm{Ln}}(U;W) = \frac{1}{\sqrt{N}\pi}\|\mathrm{Ln}(W^\dagger U)\|_F \in [0,1]$ and the span $E_{\mathrm{diam}}(U;W) = \frac{1}{2}\mathrm{diam}(\mathrm{spec}(W^\dagger U)) \in [0,1]$. For the bi-invariant metric $E_{\mathrm{Ln}}(U;W)$, see, e.g., [3]. One can compare E_{Ln} with the anisotropic metric from [4] and compare E_{diam} with the worst-case distance (16) from [5]. Both $E_{\mathrm{Ln}}(U;W)$, $E_{\mathrm{diam}}(U;W)$ are undifferentiable.

This talk is inspired, e.g., by: a) [6] which, for constructing a GRAPE-type method under PConst. coherent, incoherent controls, derives an analytical gradient under dissipative dynamics via the Wilcox formula [7], diagonalization-based approach; b) [8] which considers the Liouville–von Neumann equation, PConst. controls, construct a regularized adaptation of the Newton–Raphson method via the Van Loan method [9], etc.; c) [10] which defines controls in the terms of certain Chebyshev-based wavelets.

2. Class of controls, dynamic system. For this talk, the author operates with the following Chebyshev-inspired class of PConst. controls (not the wavelets' level):

$$\Big\{u_l(\tau) = \sum\nolimits_{j=0}^{M-1} \theta_{[\tau_j,\tau_{j+1})}(\tau) \sum\nolimits_{n=0}^{K} c_{l,n}\varphi_{n,j}^{\text{QR-Scaled}},\ |u_l(\tau)| \le \nu,\ \tau \in [0,1],\ l = \overline{1, N_u},$$

$$\varphi_{n,j} = \tau_{j,\mathrm{mid}}^{q_L}(1-\tau_{j,\mathrm{mid}})^{q_R}\cos(n\arccos(2\tau_{j,\mathrm{mid}}-1)),\ \mathbf{c} = (c_{l,n}) \in [-\mu,\mu]^{N_u\,(K+1)}\Big\} \quad (1)$$

where $q_L, q_R \in [1,2]$ and, as a variant, one can set $\nu = +\infty$. Here the weighted shifted first-kind Chebyshev polynomials are taken at a given uniform grid over $[0,1]$, "QR-Scaled" means the QR-decomposition (Q) followed by max-entry scaling. $u(\Delta\tau/2) = u(1-\Delta\tau/2) \approx 0$ is needed. This talk considers unitary (non-dissipative) dynamics in $SU(N)$ driven by (1):

$$\frac{dU^{(\mathbf{c},t_f)}(\tau)}{d\tau} = -i\,t_f\Big(H_0 + \sum\nolimits_{l=1}^{N_u} V_l u_l(\tau)\Big)U^{(\mathbf{c},t_f)}(\tau),\quad U^{(\mathbf{c},t_f)}(0) = I_N,\quad \tau \in [0,1], \quad (2)$$

where the N-level Schrödinger equation is considered with a given traceless Hamiltonian, H_0, $V_1, ..., V_{N_u}$ are constant matrices, t_f relates to t in $t = \tau t_f$, $\tau \in [0,1]$. In contrast to [11,6,8], this talk operates with (1), therefore does not use the TV-Huber regularizer taken in [11] for eliminating spectral noise of "pure" piecewise constant controls.

3. Smooth problem. Dual Annealing (DA) + Newton–Levenberg–Marquardt + Polyak. For (1), (2), take, e.g.,

$$f_1(\mathbf{c},\xi)=E_{\rm abs}(U^{(\mathbf{c},t_f)}(1);W)+\beta_{\rm Tikh.}\,t_f\,\mathbf{c}^{\mathsf T}Q_{\rm Tikh.}\mathbf{c}+\beta_{\rm Sob.}\,t_f^{-1}\,\mathbf{c}^{\mathsf T}Q_{\rm Sob.}\mathbf{c}+\beta_\xi\xi+\beta_g g(\mathbf{c},\xi)\to\min.$$

Here $\xi=\ln t_f$ and $g(\mathbf{c},\xi)$ is a certain C^3-differentiable Smoothed Augmented Lagrangian Function (C^3-SLAF) used for determining some local trust region in $[-\mu,\mu]^{N_u\,(K+1)}\times[\underline{\xi},\overline{\xi}]$. Thus, $\beta_g=0$ at the global stage in DA and $\beta_g>0$ at the local stage in DA. (DA is well known, combines the Generalized Simulated Annealing method (global zeroth-order stochastic search) and a local minimizer.) Moreover, the author adds C^3-SLAF for the constraint $|u_l(\tau)|\le\nu$. The adaptation of the Newton–Levenberg–Marquardt method ($H+vD$) [12] for the local stage is based on deriving $\nabla f_1(\mathbf{c},\xi)$, $H=\nabla^2 f_1(\mathbf{c},\xi)$ (using [7,9,13]) and supported by the Gershgorin theorem, Cholesky and QR decompositions, and Armijo line search. Taking into account [14], etc., the Polyak's momentum and its restarts are added. The notion of Morse index [15] is involved. The traceless Hamiltonian [16, (6.16)] and $\mathrm{proj}_{SU(N)}$(fSim) are used. Moreover, $E_{\rm Ln}$, $E_{\rm diam}$ can be seen posteriori taking into account $Z(SU(4))\cong\mathbb{Z}_4$. Some corresponding numerical results computed by the author will be shown in the talk.

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A Special Case of Limit Cycle Existence in a Piecewise Linear Second-order Differential Equation with a Parabolic Switching Line

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The paper investigates the behavior of solution trajectories for piecewise linear second-order differential equations with a discontinuous right-hand side [1] and a parabolic switching line. The points where the trajectories are tangent to the switching line are referred to as contact points [2]. For the zero contact point at the vertex of the parabola, previous conditions in the form of strict inequalities were derived in [3] to ensure the emergence of a stable or unstable focus. In the case of the instability and under certain additional constraints, the existence of a limit cycle was established, representing a manifestation of nonlinear effects. The present work focuses on the boundary case where these strict inequalities vanish into a single equality. An improved asymptotic representation of the solution near the zero contact point has been obtained. New conditions for the coefficients have been derived, under which a limit cycle also emerges in this special case. These results extend and generalize the research presented in [3].

Keywords: differential equations with discontinuous right-hand side, piecewise linear systems, non-linear switching line, contact point, limit cycle

1 The main results

In the present work, we consider a piecewise-linear system of differential equations:

$$\begin{cases} x' = y, \\ y' = -g(x, y), \end{cases} \tag{1}$$

where the function $g(x, y) = g_i(x, y) \equiv a_i y + b_i(x - c_i)$ if the point $(x, y) \in G_i$, with the regions defined as $G_i = \{(x, y) : (-1)^i h(x, y) > 0\}, i = 1, 2$. The coefficients of the right-hand sides satisfy the conditions $a_i^2 < 4b_i$ and $a_i \neq 0$ for $i = 1, 2$. The switching line is defined by the equation $h(x, y) = y^2 - x = 0$.

For the switching system (1), the following Theorem has been proved, which serves as a supplement to Theorem 1 in [3].

Theorem 1 (Supplement). *Let the coefficients in system* (1) *satisfy the conditions:* $a_i^2 < 4b_i$, $a_i \neq 0$, $i = 1, 2$, $c_1 < 0 < c_2$, *and* $2b_2c_2 < 1$. *Then the contact point* $(0, 0)$ *for the piecewise linear system is a stable focus if* $P_1 > P_2$, *and an unstable focus if* $P_1 < P_2$. *In the case of equality* $P_1 = P_2$, *the contact point* $(0, 0)$ *is a stable focus if* $K_1 > K_2$, *and an unstable focus if* $K_1 < K_2$, *where* $P_i = a_i/(b_ic_i(2b_ic_i - 1))$ *and* $K_i = (2a_i^3(54b_i^2c_i^2 - 9b_ic_i + 11) + 9a_ib_i(2b_ic_i - 1)(20b_i^2c_i^2 - 7b_ic_i + 1))/(b_i^3c_i^3(2b_ic_i - 1)^3)$ *for* $i = 1, 2$.

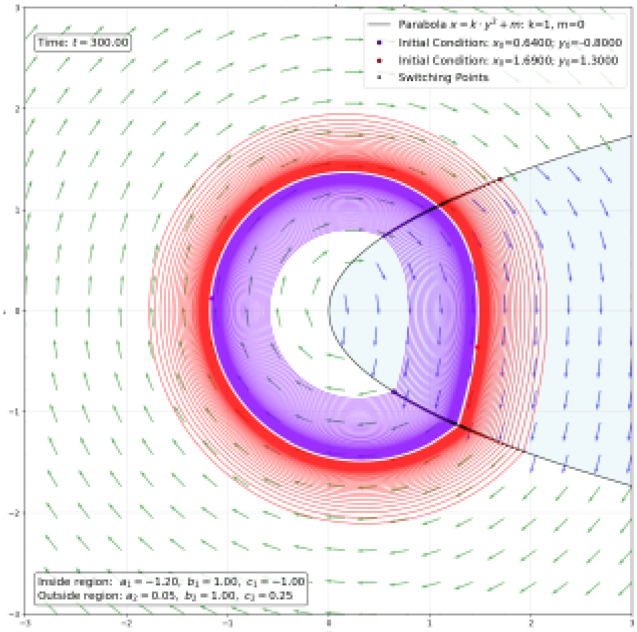

Figure 1: Convergence of system trajectories to a limit cycle under the conditions of Theorem 2.

To prove Theorem 1, we employed an asymptotic expansion in terms of a small parameter $\varepsilon > 0$, obtained for the ordinate $y(t(\varepsilon))$ of the intersection point with the lower branch of the parabola of the solution trajectory for system (1). This trajectory was initiated in both time directions from the point $(\varepsilon^2, \varepsilon)$, located on the upper branch of the parabola in a small neighborhood of the zero contact point.

For the case of instability at the zero contact point the following theorem has been proved.

Theorem 2. *Let: (a) the coefficients in system* (1) *satisfy the conditions* $a_i^2 < 4b_i$, $(i = 1, 2)$, $c_1 < 0 < c_2$, $a_2 > 0$; *(b)* $D_i \equiv a_i^2 + 2b_i(2b_i c_i - 1) \leq 0$, $(i = 1, 2)$; *(c)* $P_1 = P_2$; *(d)* $K_1 < K_2$. *Then the singular point* $(0, 0)$ *is an unstable focus for the switching system* (1), *and a limit cycle exists for the solution trajectories.*

The proof of this theorem is constructive and based on the construction of a sequence $\{x(t_{2i})\}$ of intersection points between the solution trajectory of system (1) and the upper branch of the parabola. This trajectory was initiated from a small neighborhood of the zero contact point. Under the conditions of the theorem, this sequence is monotonically increasing and bounded from above, which implies its convergence. The existence of a limit cycle follows directly from this convergence. An example of the emergence of a limit cycle is shown in Fig. 1.

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Study of the Oscillatory Flow Regime of a Surfactant Solution in an Annular Channel*

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The real anomalous temperature dependence of viscosity for a micellar solution of the cationic surfactant EHAC and the binding salt SHNC was considered. Oscillatory regimes of flow rate change of anomalously thermoviscous liquids over time were discovered.

Keywords: thermoviscous liquid, annular channel, viscous barrier, flow rate

1 The main results

The flow of a micellar surfactant solution [1] in an annular channel is considered. It is assumed that the temperature of the inflowing liquid is higher than the ambient temperature. Figure 1 shows the viscosity-temperature curve. The black circles represent the data for the cationic surfactant solution from [1], and the color curve represents their approximation.

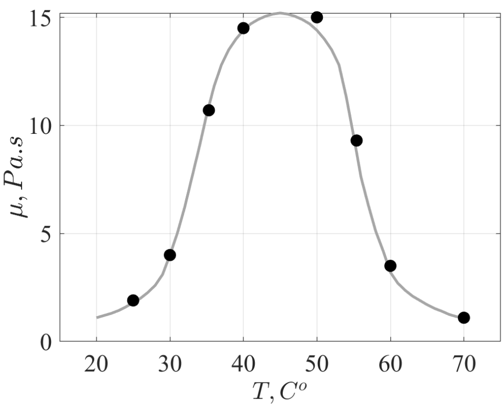

Figure 1: Dependence of viscosity on temperature

The mathematical model consists of the continuity, Navier-Stokes, and energy conservation equations, written in a cylindrical coordinate system taking into account axial symmetry in dimensionless form. The system of equations of the mathematical model was solved numerically using the control volume method and the SIMPLE (Semi-Implicit Method for Pressure-Linked Equation) algorithm, modified to account for a variable viscosity coefficient. The original computer code is implemented in C++ in the cross-platform Qt Creator development environment.

The influence of boundary conditions on the flow structure and fluid flow rate is demonstrated. Both weakly nonlinear Thomson-type oscillations and damped oscillations were

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obtained. The main patterns clearly confirm the results of studies [2]. The influence of boundary conditions on the flow structure and fluid flow rate is demonstrated. Characteristic flow regimes and the factors determining their stability are identified.

The research was supported by the state budget funds for the state assignments FMRS-2024-0001 and FMRS-2026-0012.

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Sturm Sequences for Delay-Independent Stability

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We consider absolute asymptotic stability of linear delay differential systems with one concentrated delay. We reduce the problem to checking stability of the zero-delay polynomial and positivity of an auxiliary polynomial obtained from a resultant. The latter condition is verified by Sturm sequences, which yields an effective algebraic test for delay-independent stability.

Keywords: delay differential equation, absolute asymptotic stability, resultant, Sturm sequence

1 Results

Consider the system

$$\dot{x}(t) + Ax(t) + Bx(t-h) = 0, \qquad t \geq 0, \tag{1}$$

where $h > 0$ and $A, B \in \mathbb{R}^{N \times N}$. Its characteristic function is

$$\Phi_h(z) = \det(Iz + A + Be^{-hz}).$$

System (1) is called *absolutely asymptotically stable* if it is asymptotically stable for every $h > 0$.

The study of delay-independent stability for such systems goes back at least to the early work of Repin [1]. To investigate purely imaginary roots of Φ_h, define

$$\Lambda_z(u) = \det(Iz + A + Bu), \qquad F_\omega(u) = u^m \Lambda_{i\omega}(u^{-1}), \qquad F^*_\omega(u) = \Lambda_{-i\omega}(u),$$

where $m = \deg\det(A + Bu)$, and let

$$\mathcal{R}(\omega) = \mathrm{Res}_u(F_\omega, F_\omega^*).$$

The resultant $\mathcal{R}(\omega)$ is a real even polynomial, hence there exists a real polynomial P such that $\mathcal{R}(\omega) = P(\omega^2)$.

A key fact is that the following statements are equivalent:

1. there exist $h \geq 0$ and $\omega \in \mathbb{R}$ such that $\Phi_h(i\omega) = 0$;
2. there exist $\omega \in \mathbb{R}$ and $u \in \mathbb{C}$, $|u| = 1$, such that $F_\omega(u) = F_\omega^*(u) = 0$.

Therefore, if Φ_0 is stable and $P(s)$ has no nonnegative real roots, then system (1) is absolutely asymptotically stable.

Thus, absolute asymptotic stability is reduced to two finite-dimensional tests:

1. stability of Φ_0;
2. absence of nonnegative real roots of $P(s)$.

To verify positivity of P on $[0, +\infty)$, we use Sturm sequences. Define recursively

$$P_0 = P, \quad P_1 = P', \quad P_{k-1} = Q_k P_k - P_{k+1}, \quad \deg P_{k+1} < \deg P_k,$$

that is, P_{k+1} is the negative remainder after division of P_{k-1} by P_k. This process yields the Sturm chain

$$P_0,\ P_1,\ P_2, \dots, P_m.$$

By Sturm's theorem [2], for $a < b$ such that $P(a)P(b) \neq 0$, the number of distinct real roots of P on (a, b), counted without multiplicities, equals $V(a) - V(b)$, where $V(x)$ is the number of sign changes in the Sturm chain at x. We also write

$$V(+\infty) = \lim_{x \to +\infty} V(x).$$

In particular, if $P(0) > 0$ and $V(0) = V(+\infty)$, then P has no roots on $(0, +\infty)$.

Example

Consider the matrices

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\Phi_h(z) = \det\bigl(Iz + A + Be^{-hz}\bigr) = z^2 + 3z + 2 - ze^{-hz} - e^{-2hz}.$$

For $h = 0$, we obtain

$$\Phi_0(z) = z^2 + 2z + 1 = (z + 1)^2,$$

which is stable.

In this case, the corresponding resultant is

$$\mathcal{R}(\omega) = \omega^8 + 9\omega^6 + 24\omega^4 + 29\omega^2 + 9.$$

Hence

$$P(s) = s^4 + 9s^3 + 24s^2 + 29s + 9.$$

Its Sturm sequence starts with $P_0(s) = P(s)$, and

$$P_1(s) = 4s^3 + 27s^2 + 48s + 29, \; P_2(s) = \frac{51}{16}s^2 + \frac{21}{4}s + \frac{117}{16},$$
$$P_3(s) = -\frac{1504}{289}s + \frac{5152}{289}, \qquad P_4(s) = -\frac{554013}{8836}.$$

At $s = 0$ the signs are $+, +, +, +, -$, so $V(0) = 1$.

As $s \to +\infty$, the signs are $+, +, +, -, -$, hence $V(+\infty) = 1$. Since $P(0) = 9 > 0$, Sturm's theorem implies that P has no roots on $(0, +\infty)$.

Therefore, $P(s)$ has no nonnegative real roots. Since Φ_0 is stable, system (1) is absolutely asymptotically stable.

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Numerical Simulation and Performance Evaluation of Optimal Variable Order Fractional PID Controller by Variations of Variable Order Fractional Calculus

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We propose an optimal variable order fractional PID controller design method by particle swarm optimization(PSO) algorithm, and evaluate the performance comparison of PID, Fractional PID(FPID) and Variable-order Fractional PID(VF-PID) controllers by the variations of variable order fractional calculus via MATLAB simulation of two examples. The performance of the B-type variable order fractional controller is shown to be the best.

Keywords: variable order fractional PID controller, variable order fractional dynamic system, variable order fractional calculus, PSO algorithm

Problem formulation

In this section, we propose the optimal variable order fractional P$I^{\beta(t)}D^{\alpha_0(t)}$ controller design method for multi-term fractional dynamic system with time-varying coefficients. The basic governed dynamic equation that we consider is the multi-term variable-order fractional nonhomogeneous linear differential equation with the time-varying coefficients.

$$^{C}D_{0+}^{\alpha(t)}y(t) + \int_{i=1}^{L} a_i\left(t\right) {}^{C}D_{0+}^{\alpha_i(t)}y\left(t\right) = u\left(t\right) \qquad 0 < t < T \tag{1}$$

subject to the initial condition

$$D^k y(t)\,|_{t=+0} = b_k, \quad k = 0, 1, \dots, n-1 \tag{2}$$

whereT > 0, $\alpha(t)$, $\alpha_i(t) \in C_{[0,T]}$, i = 1,..., L, $n-1 < \alpha(t) \le n$, $n_i - 1 < \alpha_i(t) \le n_i$, n, $n_i \in N$ are the known functions as variable order, $a_i(t)$ are the known continuous functions as time-varying coefficients. Throughout the paper, we assume the existence and uniqueness of the solution. The integral of time-weighted absolute error is the criteria function used for identifying the optimal value of controller parameters.

$$ITAE(parameters) = \int_0^\infty t|e(t, parameter)|dt$$

Optimal variable order fractional PID controller design

The feedback control signal by the P$I^{\beta(t)}D^{\alpha_0(t)}$ controller for this dynamic system is as follows:

$$u_b(t) = k_p e(t) + k_{i0} I_t^{\beta(t)} e(t) + k_{d0} D_t^{\alpha_0(t)} e(t) \tag{3}$$

The feedforward control signal is

$$u_f(t) = {}^cD_{0+}^{\alpha(t)} x_d(t) + \int_{i=1}^{L} a_i(t)\,{}^cD_{0+}^{\alpha_i(t)} x_d(t) \tag{4}$$

The control input signal is $u(t) = u_f(t) + u_b(t)$ and the desired state (or designed trajectory) is $x_d(t)$.

Then, the dynamic equation of the closed-loop control system for the control object is expressed as

$$\begin{cases} e(t) = x_d(t) - y(t) \\ {}^cD_{0+}^{\alpha(t)} e(t) + \int_{i=1}^{n} a_i(t)\,{}^cD_{0+}^{\alpha_i(t)} e(t) = k_p e(t) + k_{i0} I_t^{\beta(t)} e(t) + k_{d0} D_t^{\alpha_0(t)} e(t) \end{cases} \tag{5}$$

where $e(t)$ is the following error signal as the solution of (5), k_P, k_i and k_d represent coefficients of proportional gain, integral and derivative operators and (t), $\alpha_0(t)$ are arbitrary positive functions of time with restricted image. If we uniformly divide [0, T] by $0 = t_0 < t_1 < t_2 < \dots < t_N = T$, $t_m = mh$, $m = 0, \dots N$, (5) leads to the following left A-type, B-type and C-type backward finite difference equations.

By considering two examples when the highest differential orders of the dynamic system and the feedback controller are $2 < \alpha(t) \le 3$ and $1 < \alpha(t) \le 2$, we simulate some dynamic systems with PID, FPID, VFPID controllers by variable order fractional derivative and integral of types A, B and C.

Algorithm for designing optimal VFPID controller by PSO

Step 1. Initialize a group consisted of particles with the dimension of VFPID parameters.
Step 2. Design VFPID for each particle and get the solution e(t).
Step 3. For each particle, estimate the fitness (ITAE) and find the best particle.
Step 4. Update velocity and position of particles.
Step 5. Repeat from step 2 to step 4 until the maximum number of iteration.
Step 6. Output the best particle and its VFPID.

Conclusion

In this paper, we considered a numerical method for solving multi-term time-varying linear fractional differential equation with variable order by approximating to the finite difference equation using the Grünwald-Letnikov fractional derivative formula. This method can be generalized to numerical solution of multi-term nonlinear fractional differential equation.

Based on this, the optimal integer-order PID, fractional PID and variable-order fractional PID controller design methods for fractional dynamic system are proposed using particle swarm optimization algorithm. The optimal parameter identification and performance comparison of various types of controllers show that VFPID is superior to FPID and FPID is superior to PID through two numerical experiments with MATLAB. Quantitatively, VFPID (average value of VFPID) outperforms FPID by 1.46 times and PID by 4.514 times, while FPID outperforms PID by 3.085 times. In particular, VFPIDBB is 6.54 times more than FPID and 20.2 times more than PID. Also, if load disturbance and coefficient uncertainty are added to the system simultaneously, the PID, FPID, VFPIDAA and VFPIDBB controllers have an average variation of 28.44%, 20.56%, 19.97%, and 6.348% and the suppression and robustness are enhanced from integer to variable order fractional. we studied found that the controller with the B-type variable-order fractional calculus showed very superior control performance, which motivated interest in its analog circuits, device implementation and its application.

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Estimates of Solutions to Nonlinear Systems with Periodic Coefficients in Linear Terms and Several Ddelays

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We consider systems of nonlinear differential equations with several delays and periodic coefficients in linear terms. The aim is to obtain estimates characterizing decay rates of solutions at infinity and estimates for attraction sets of the zero solution. A special Lyapunov–Krasovskii functional is used.

Keywords: time-delay systems, Lyapunov–Krasovskii functional, stability, exponential decay

We consider nonlinear systems of delay differential equations with periodic coefficients in linear terms of the form

$$\frac{d}{dt}y(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(t-\tau_j) + F(t, y(t), y(t-\tau_1), \dots, y(t-\tau_m)), \qquad (1)$$

where $t > 0$, $A(t)$ and $B_j(t)$, $j = \overline{1,m}$, are $n \times n$ matrices with continuous T-periodic entries, τ_j, $j = \overline{1,m}$, are delays, $0 < \tau_1 < \tau_2 < \dots < \tau_m$. The real-valued continuous vector function $F(t,u) = F(t, u_0, u_1, \dots, u_m)$ is Lipschitz in u_0 on every compact set $G \subset [0,\infty) \times \mathbb{R}^{n(m+1)}$.

The exponential stability of the zero solution to the linear systems ($F(t,u) \equiv 0$) was studied in [1] by using the modification of the Lyapunov-Krasovskii functional proposed in [2]

$$\langle H(t)y(t), y(t)\rangle + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} \langle K_j(t-s)y(s), y(s)\rangle ds, \tag{2}$$

where $H(t)$ and $K_j(s)$ are matrices such that

$$H(t) = H^*(t) > 0, \quad H(t) = H(t+T), \quad K_j(s) = K_j^*(s) > 0, \quad j = 1, \dots, m.$$

We assume that the vector function $F(t,u)$ defining nonlinear terms in (1) satisfies the estimate

$$2\langle H(t)F(t, u_0, \dots, u_m), u_0\rangle \le \langle H(t)u_0, u_0\rangle \sum_{k=0}^{m} q_k \|u_k\|^{\omega_k},$$

$$q_k,\ \omega_k > 0, \quad k = 0, 1, \dots, m, \quad F(t,0) = 0.$$

Using the functional (2), we establish estimates of exponential decay of solutions to (1) at infinity and obtain estimates for attraction sets of the zero solution. This work continues our investigations of the exponential stability of the zero solution of systems of the form (1) in the case of constant coefficients [3].

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Spectral Properties of Sturm-Liouville Type Operators

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This work is devoted to the study of the fundamental problems of the spectral theory of Sturm–Liouville type differential operators. In this study, the classical Sturm–Liouville theorem is proved, which characterized the structure of spectral

operators. The influence of various types of boundary conditions on the spectral properties is briefly examined. The results obtained in the course of the study provide a theoretical foundation for the investigation of more complex boundary value problems in future research.

Keywords: Sturm–Liouville, spectral theory, eigenvalue, asymptotic

1 The main results

The classical Sturm–Liouville operator is given by the following differential expression:

$$Ly = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y$$

Where $p(x)$ denotes a function that satisfies condition $p(x) > 0$, while $q(x)$ is a given real-valued function. The operator, together with the corresponding boundary conditions, is defined in the Hilbert space $L^2\ (a,b)$. In this study, the following eigenvalue problem is considered:

$$Ly = \lambda y,$$

here λ denotes the spectral parameter.

Theorem 1. ***Theorem.*** *(Classical Sturm–Liouville theroem.) Under appropriate boundary conditions, a Sturm–Liouville-type operator is self-adjoint, and its spectrum is real, discrete, and bounded below. The eigenvalues of the operator form an increasing sequence of the form*

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots,\ \lambda_n \to +\infty$$

And the corresponding eigenfunctions constitute an orthonormal system in the space $L^2\ (a,b)$.

This theorem is one of the fundamental results characterizing the spectral structure of Sturm–Liouville operators. As a direct consequence of the theorem, important results concerning the discreteness of the spectrum and the distribution of eigenvalues are obtained. According to well-known classical results, for sufficiently large n, the eigenvalues λ_n satisfy the following asymptotic relation:

$$\lambda_n = (n\pi/(b-a))^2,\ n \to \infty.$$

This asymptotic formula characterized the distribution of the spectrum. Such results are among the fundamental theorems of spectral theory and are widely applied in solving problems of mathematical physics.

In addition, the influence of various types of boundary conditions on the spectral properties has been briefly investigated.

As a results of the conducted research, the main principles of the spectral theory of Sturm–Liouville type operators have been systematized, and the significance of these operators in functional analysis and mathematical physics has been euphuized. The obtained results provide a theoretical foundation for the future study of non-classical Sturm–Liouville operators, and more complex boundary value problems.

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On the Regularity of the Solution to one Uniformly Strictly Pseudohyperbolic Equation*

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A class of strictly pseudohyperbolic equations of the fourth order with variable coefficients is considered. This class includes, in particular, the Vlasov, Gal'pern, and Rayleigh–Bishop operators. Under specific conditions for data, estimates are established.

Keywords: pseudohyperbolic equation, energy estimates, weighted Sobolev spaces, regularity of the solution

We consider the differential equation of the fourth order of the following form

$$L(x; D_t, D_x)u$$
$$= (aI - \sum_{|\beta|=2} a^0_\beta(x) D^\beta_x) D^2_t u + \sum_{|\beta|=3} a^1_\beta(x) D^\beta_x D_t u + \sum_{|\beta|\le 4} a^2_\beta(x) D^\beta_x u = f(t,x), \quad a > 0, \tag{1}$$

where the coefficients are real-valued sufficiently smooth functions, which are constant outside of some ball. We suppose that the operator $L(x_0; D_t, D_x)$ at any fixed point x_0 is strictly pseudohyperbolic (see [1,4]).

Note that pseudohyperbolic equations arise when studying some problems of hydrodynamics, elasticity theory, when constructing waveguides, etc. (see, for example, [5–7]).

Theorem [8]. *Let the operator $L(x; D_t, D_x)$ is uniformly strictly pseudohyperbolic. Then there is a constant $\gamma_0 > 0$, and for any $f(t,x) \in W^{k,q}_{2,\gamma}(R^{n+1}), \gamma > \gamma_0$, the solution of equation (1) belongs to the space $W^{1+k,3+q}_{2,\gamma}(R^{n+1})$, and the following estimate holds*

$$\begin{gathered} \|u(t,x), W^{1+k,3+q}_{2,\gamma}(R^{n+1})\| + \sum_{|\alpha|=2+q} \|D^{1+k}_t D^\beta_x u(t,x), L_{2,\gamma}(R^{n+1})\| \\ \le c(\gamma) \|f(t,x), W^{k,q}_{2,\gamma}(R^{n+1})\|, \end{gathered} \tag{2}$$

where $c(\gamma) > 0$ is a constant that does not depend on $f(t,x)$.

Estimate (2) is an analogue to the estimate for strictly hyperbolic equations with constant coefficients [2-4].

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Numerical Method for Solving Time-Space Fractional Stefan Problem by Using APT Algorithm

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In this paper, a numerical method for solving the two-phase time-space fractional Stefan problem using an alternating phase truncation method is presented. For fractional differential equations with fractional derivatives both in terms of time and space variables, an alternate phase truncation method using the enthalpy distribution in the region under consideration is used to determine the moving boundary between the solid and liquid phases with an approximate temperature distribution in the region.

Keywords: fractional stefan problem, alternating phase truncation method, enthalpy distribution, fractional derivative

1 Introduction and problem formulation

We consider the following one-dimensional two-phase fractional Stefan problem. The considered area consists of two sub-regions, namely the liquid phase region $_L$ and the solid phase region Ω_S.

$$c_L \rho_L {}^c D_t^\alpha T_L\left(x,t\right) = \hat{\lambda}_{L\,RL} D_x^\beta T_L\left(x,t\right), b_1 < x < s(t), t \in (0,t^*) \tag{1}$$

$$c_S \rho_S {}^c D_t^\alpha T_S(x,t) = \hat{\lambda}_{S\,RL} D_x^\beta T_S(x,t), s(t) < x < b_2, t \in (0,t^*) \tag{2}$$

where c_k- specific heat([J/(kgK)]), ρ_k- density([kg/m^3]), $\hat{\lambda}_k = \hat{w}\lambda_k$- the scaled thermal conductivity ([W/($s^{\alpha-1}$mK)]), that is the thermal conductivity multiplied by the scaling constant $\hat{w}$ with numerical value of one and units [$s^{\alpha-1}$] chosen such that the right and left units of the equation are the same, λ_k- thermal conductivity([W/(mK)]), where index $k = S$ denotes the solid phase and $k = L$the liquid phase. We assume that the values of the parameters $c_S, \rho_S, \hat{\lambda}_S, c_L, \rho_L, \hat{\lambda}_L$ are constant.

In the two subregions, the time derivative of the temperature distribution is the Caputo fractional derivative of order $\alpha(0 < \alpha < 1)$and the derivative with respect to the spatial variable is the Riemann-Liouville fractional derivative of order $\beta(1 < \beta < 2)$. The initial and boundary conditions are as follows:

$$T_S(x, 0) = T_0(x), b_1 \leq x \leq b_2, \tag{3}$$

$$T_k(b_1, t) = T_z(t), t \in (0, t^*), k \in \{L, S\}, \tag{4}$$

$$T_k(b_2, t) = T_b(t), t \in (0, t^*), k \in \{L, S\}. \tag{5}$$

The conditions of temperature continuity and Stefan condition at the moving boundary separating the phases are given by

$$T_S(s(t), t) = T_L(s(t), t) = T_m, \tag{6}$$

$$-\overline{\lambda}_L \frac{\partial T_L(x,t)}{\partial x}\bigg|_{x=s(t)} = -\overline{\lambda}_S \frac{\partial T_S(x,t)}{\partial x}\bigg|_{x=s(t)} + L\rho S^C D_t^\alpha S(t) \tag{7}$$

where L is the latent heat of fusion by unit of mass ([J/kg]) and T_m([K]) denotes the temperature of the phase transition.

2 Numerical Method using APT Algorithm

The Alternating phase truncation method was first proposed in [1]. In [2], the two-phase Stefan problem was solved using this method.Introducing one-side limits A and latent heat k, the considered model of the Stefan problem(1)-(7) in the enthalpy convention for $t \in (0, t^*)$ takes the form

$$\begin{gathered}
{}^c D_t^\alpha H_L(x,t) = \hat{\lambda}_{L\,RL} D_x^\alpha H_L(x,t) + c_L\rho_L\left(G(x,t) - \hat{\lambda}_L F(x)\right), b_1 < x < s(t) \\
{}^c D_t^\alpha H_S(x,t) = \hat{\lambda}_{S\,RL} D_x^\beta H_S(x,t) + c_S\rho_S\left(G(x,t) - \hat{\lambda}_S F(x)\right), s(t) < x < b_2 \\
-\hat{a}_L \frac{\partial H_L(x,t)}{\partial x}\bigg|_{x=s(t)} = -\hat{a}_S \frac{\partial b H_S(x,t)}{\partial x}\bigg|_{x=s(t)} + k^c D_t^\alpha s(t) \\
A_2 = A_1 + k \\
H_S(x,0) = H(T_0(x)), b_1 \leq x \leq b_2 \\
H_k(x,t)|_{x=b_1} = H(T_z(t)), k \in \{L, S\} \\
H_k(x,t)|_{x=b_2} = H(T_b(t)), k \in \{L, S\}
\end{gathered} \tag{8}$$

In the domain of space, we introduce the grid $x_i = ix, i = 0, 1, \cdots, N$ and in the domain of time, the grid: $t_j = jt, j = 0, 1, \cdots, K$. For simplicity, from now on we will omit the index

$k \in \{S, L\}$. Belonging to the proper phase will be identified basing on the value of the enthalpy in the given point. Hence, we introduce the notation $H(x_i, t_j) = H_i^j$.

By introducing the following notation

$$\sigma(\alpha, \Delta t) = \frac{1}{\Gamma(1-\alpha)(1-\alpha)(\Delta t)^\alpha} \qquad \omega(\alpha, r) = r^{1-\alpha} - (r-1)^{1-\alpha}$$

the Caputo fractional derivative of the enthalpy with respect to the time variable is approximated by

$$^cD_t^\alpha H_i^j = \sigma(\alpha, \Delta t) \sum_{r=1}^{j} \omega(\alpha, r)(H_i^{j-r+1} - H_i^{j-r}) \tag{9}$$

where $\alpha \in (0, 1)$.

On the other hand, the Riemann-Liouville fractional derivative of the enthalpy with respect to the spatial variable is approximated by Second-order compact approximation.

$$_{RL}D_x^\beta H_i^j = \frac{1}{(\Delta x)^\beta} \sum_{k=0}^{i} \omega_k^\beta H_{i-k+1}^j, \tag{10}$$

Using (9)and (10), we can formulate the equation of the heat conduction in the discrete form

$$\sigma(\alpha, \Delta t) \sum_{r=1}^{j} \omega(\alpha, r)(H_i^{j-r+1} - H_i^{j-r}) = \frac{\hat{\lambda}}{(\Delta x)^\beta} \sum_{k=0}^{i} \omega_k^\beta H_{i-k+1}^j + c\rho(G_i^j - \hat{\lambda} F_i).$$

This is the heat conduction equation for the internal nodes $i = 1, 2, \cdots, N-1$.

Therefore, the error of the scheme is of order $O(\Delta t^{2-\alpha} + \Delta x^2)$. Due to application of the implicit scheme, it is unconditionally stable.

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Preparation of Tubular Ceramic Membranes using Kaolin and Purification of Ginkgo Biloba Leaf Flavonoid

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Tubular ceramic membranes with SiO_2 separation layer were prepared using kaolin mineral as support and intermediate layer, and the separation and concentration of flavonoids in Ginkgo biloba leaf extract was carried out to evaluate the feasibility of membrane treatment. The sintering temperature, retention time and amount of kaolin mineral that affects the average pore size, porosity and compressive strength of ceramic membrane support were chosen as factors and optimized design was carried out with the aim of increasing pore size, porosity, compressive strength and reducing surface roughness. The average pore diameter of the intermediate layer prepared using kaolin is 100nm and the ultrafiltration membrane, which formed a γ-Al_2O_3 layer by the sol-gel method on the intermediate layer, has a pore size of 5nm and the membrane with a SiO_2 layer formed on the ultrafiltration membrane layer was used as a nanofiltration membrane with a pore size of 1nm. The results of the experiments on the separation of flavonoids from extracts showed that the separation of flavonoids from Ginkgo biloba extracts was achieved at 30L/m 2 h ¦bar in ultrafiltration membrane with 5nm pores and 60% concentration in nanofiltration membrane with 1nm pore size.

Today, the main focus of research on inorganic membranes is on the reduction of membrane performance and production cost.

In this paper, an optimization design was carried out to determine the optimum conditions for the support structure and to prepare different pore size separation membranes to separate and purify flavonoids from extracts of Ginkgo biloba leaves, with the aim of improving pore size, porosity, compressive strength and reducing surface roughness, using low-cost domestic kaolinite.

Kaolinite was obtained from Sammyo-ri, about 20km from the west of the Kangso District in the western central part of the DPRK. This clay contains 90% kaolinite and 10% illite. The support forming was done with a single screw extruder (Brabender, Germany). The final heat treatment temperature was 2 h according to the heat treatment program at 1250°C, and the geometric dimensions were ID = 10mm and OD = 15mm. Surface roughness of the support was measured using a measuring device (visolator). The pore size of each sample of the support was measured using the bubble method with an open pore apparatus.

The mechanical strength of the sintered samples was measured by compression tests at a constant displacement rate of about 0.2 mm/min using a mechanical testing machine (Zwick 10K) (Germany). The compressive strength was determined by dividing the force exerted on the specimen by the cross-sectional area of the specimen when the specimen failed. The porosity of the tubular ceramic membrane support was determined by Archimedes method.

To study the effect of binder (kaolinite mineral powder) on the properties of ceramic membrane support, we used a mixed powder with kaolin mineral as binder. Since the filler is more than 60% of the feed, the binder was selected as 15wt.%, 25wt.% and 30wt.% of the total feed powder.

To investigate the effect of sintering temperature on the porosity characteristics and crystalline phase of the support, the sintering temperature was first tested at 1100 °C, 1501 °C, 1200°C, 1250°C and 1300°C, respectively, and the holding time was 2h.

Based on the above experimental data, the retention time was set at 2h to increase the pore size, porosity and compressive strength from the membrane support requirements.

In this paper, an optimal design was carried out with the aim of increasing pore size, porosity, compressive strength and reducing surface roughness using Ansys 14.5.
As described earlier, the optimum design variables are the amount of kaolin mineral binder added to alumina powder, sintering temperature and holding time, which can be written as follows :
Optimization variable: X=$(C_d,\ C_T,\ C_t)$
$C_{d:}$ grain: addition of binder
C_T: Sintering temperature
C_t: retention time
The objective function for increasing the pore size is

$$Max : f_1(X) = A_p(X) \tag{1}$$

Here, $A_p(X)$ is the pore size according to the design variables.
The objective function for increasing porosity is

$$Max : f_2(X) = A_r(X) \tag{2}$$

where $A_r(X)$ is the porosity with respect to the design variables.
The objective function for strength enhancement is

$$Max : f_3(X) = A_s(X) \tag{3}$$

where A is the mechanical strength according to the design variables.
The objective function for lowering the surface roughness is

$$Min : f_4(X) = A_R(X) \tag{4}$$

We have developed a multi-objective optimization design for this optimization problem using the NSGA-II genetic algorithm (Non-dominated Sorting Genetic Algorithms), one of the multi-objective genetic algorithms. To achieve the optimal design, the experimental design of the above selected parameter parameters was carried out, the mathematical model was developed using the non-parameterization method, and the optimization was carried out using the improved multi-objective genetic algorithm (NSGA-II)[1]

Using the multiobjective genetic algorithm, the maximum allowable percentage of Pareto solution was 70% with 100 initial individuals, 100 individuals per genetic generation, 20 generations, and 100 individuals per genetic generation in the multiobjective optimization.

The optimization allowed to select three Pareto solutions, and the optimal results are shown in Table. 1.The experiments were carried out for the three optimal solutions obtained, with a pore size of 2.08 micrometer, a porosity of 34.5%, a compressive strength of 30.4 MPa, a pore size of 1.8 micrometer, a porosity of 36.7%, a compressive strength of 10.1 Mpa, and

Table 1: Optimisation results obtained optimal solution set

	Candidate Point 1	Candidate Point 2	Candidate Point 3
P1-C_L	24.6	10.7	30.1
P2-Temperature	1246	1156	1180
P3-Time	117	116	97
P4-P_size	2.08	1.8	1.9
P5-Porosity	34.5	36.7	37.2
P6-Strength	30.4	10.1	11.3
P7-Surface	25.458	-23.08	24.044

for the three methods, a pore size of 1.9 micrometer, a porosity of 37.2%, and a compressive strength of 11.3 Mpa.
Thus, from the requirements of the membrane support, the first option is the most suitable. Therefore, we chose 25wt.% binder powder for alumina powder and 1,250°C for 2h for sintering temperature and prepared the support.

The separation operating conditions and concentration effects were investigated for Ginkgo biloba leaf extracts. Fig.1 shows the permeate flux characteristics with pressure and concentration of Ginkgo biloba extract with 0.1 micrometer, 5nm Al_2O_3 and 1.5nm SiO_2 membranes.

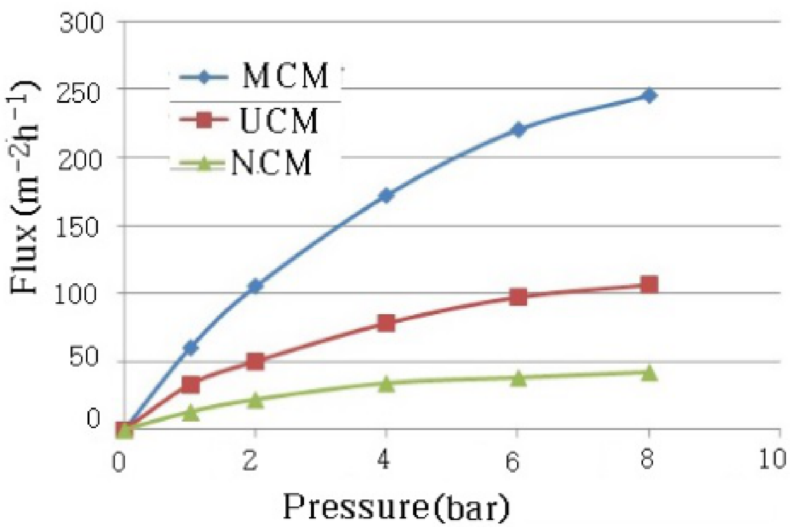

Figure 1: Ginkgo biloba leaf extract permeate of ceramic membranes with pressure.

Flavonoid separation in Ginkgo biloba leaf extracts was achieved in ultrafiltration membranes with 5nm pores, and 60% enrichment was achieved in nanofiltration membranes with 1.5nm pore sizes.
As shown in Fig. 1, the permeate of Ginkgo biloba leaf extract increased with increasing pressure applied to each membrane, and the effect gradually decreased as pore size decreased. It can be attributed to the decrease in the pollutant accumulation inside the pores due to the increase in the number of layers stacked on the support as the pore size of the membrane decreases.

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A New Relationship Between the Core and Weber Set in Games with Restricted Cooperation

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Game with restricted cooperation is a TU-game defined in the context of the existence of infeasible coalitions. We propose a new relationship in which a nonempty core has no common point with the Weber set in some games of such a context. In this paper, we were supposed to name such a relationship the strict separation between core and Weber set. We present and prove feature of the coalition structure for strict separability and sufficient conditions for strict separation under which the core is strictly separated from the Weber set in some of the games with restricted cooperation.

Keywords: game with restricted cooperation, core, Weber set, context of the existence of infeasible coalitions, strict separation between core and Weber set

1 Basic notations and Concepts

A *closure system on* N is a collection Ω of subsets of N which is closed under intersection and contains N. Denote by Q_C^i and P_C^i respectively, the last set in the sequence of chain $C \in CH(\Omega)$ which does not contain i, and the first set containing i. Then $Q_C^i \prec P_C^j$.

Definition 1 Let $(\Omega,\ v)$ be a game with restricted cooperation. The marginal vector $m^C \in R^N$ with respect to the chain $C \in CH(\Omega)$ in game $(\Omega,\ v)$ is given by

$$m_i^C := \frac{v\left(P_C^i\right) - v\left(Q_C^i\right)}{\left|P_C^i \backslash Q_C^i\right|},\ \forall i \in N \tag{1}$$

where $|\cdot|$ denotes the cardinality of $\cdot$.

Definition 2 The Weber set of a game $(\Omega,\ v)$ is the convex hull of all its marginal vectors,

$$Weber\left(\Omega,\ v\right) := conv\left\{m^C \;\middle|\; C \in CH(\Omega)\right\}$$

Definition 3 Let $\Omega \subset 2^N$ be a lattice. A nonempty subset $S \in \Omega \backslash \{\emptyset,\ \ N\}$ is a *dependent element* of Ω if $N \backslash S \in \Omega$. And a nonempty subset $S \in \Omega \backslash \{\emptyset,\ \ N\}$ is an *independent element* of Ω if S is not a dependent element of Ω. In particular, an independent element $S \in \Omega \backslash \{\emptyset,\ \ N\}$ is a *proper independent element* if each element $T \in CH_S(\Omega) \backslash \{\emptyset,\ \ N\}$ is an independent element.

In the future, we will express that the core *is strictly separated from* the Weber set unless a nonempty core has a common point with the Weber set in $(\Omega,\ v)$. Now, we will discuss some concepts.

Given a lattice $\Omega\subseteq 2^N$ and $S\in\Omega\backslash\{\emptyset,\ N\}$ we define two sub-lattices $\Omega_S:=\{T\subseteq S\mid T\in\Omega\}$ and $\Omega^S:=\{T\supseteq S\mid T\in\Omega\}$. Let $(\Omega,\ v)$ be a game with restricted cooperation. Then we can define two sub-games $v_S:\ \Omega_S\to R$ and $v^S:\ \Omega^S\to R$ as follows:

$$v_S(T):=v(T)\qquad\text{for each}\ \ T\subseteq S,\ T\in\Omega$$

$$v^S(T\backslash S):=v(T)-v(S)\qquad\text{for each}\ T\supseteq S,\ T\in\Omega$$

And let $B(N)$ be any balanced collection in Ω. Then, we can introduce the following notation.

$$\sum_{T\in B_{max}(N)}\lambda_T v(T):=\max_{B(N)}\sum_{T\in B(N)}\lambda_T v(T)\ .$$

In future, we will call $B_{max}(N)$ a *balanced collection with maximal value* of $(\Omega,\ v)$.

Definition 4 Let there exists independent elements in $\Omega\subset 2^N$. And let $S,\ T\in\Omega\backslash\{\emptyset,\ \ N\}$. A pair $(S,\ T)$ of elements is a *pair of elements with separability* if it satisfies either of two following conditions.

(1) An independent element S and an independent element T, which are not ordered from each other.

(2) A proper independent element S and the other element T, which are not ordered from each other.

Definition 5 Let $\Omega\subset 2^N$ be a lattice. Then a proper independent element $S\in\Omega$ is a *synthetic element* of Ω if $|CH_S(\Omega)|\geq 2$. Especially, a proper independent element $S\in\Omega$ is a *concatenation element* of Ω if $|CH(\Omega)|=|CH_S(\Omega)|\geq 2$.

2 The Main Results

Theorem 1. Let $\Omega\subset 2^N$ satisfy the following two conditions. Then the core is never strictly separated from the Weber set in $(\Omega,\ v)$, which is any game defined in Ω.

(1)There exists an atom or co-atom of Boolean lattice 2^N in $M=2^N\backslash\Omega$.

(2) M is a totally ordered set.

Theorem 2. Let there exist independent element in Ω such that $|CH(\Omega)|\geq 2$. And let the core is never strictly separated from the Weber set for any game $(\Omega,\ v)$ in defined in Ω. Then Ω has the following properties:

(1)The set of all the independent elements in Ω is totally ordered.

(2)For each independent element $S\in\Omega$, there exists at least one dependent element of Ω in $CH_S(\Omega)$.

Theorem 3. Let $(\Omega,\ v)$ be the game defined in $\Omega\subset 2^N$, in which there exist a pair $(S,\ T)$ of elements with separability. And let $Core(\Omega,\ v)\neq\emptyset$. If the following two conditions are satisfied for $(S,\ T)$, then the core is strictly separated from the Weber set in $(\Omega,\ v)$.

$$\sum_{i\in S}m_i^{C^*}<v(S)\,,\forall C^*\in CH(\Omega)\backslash CH_S(\Omega) \tag{2}$$

$$\sum_{i\in S}m_i^{C^*}<v(S)\,,\forall C^*\in CH(\Omega)\backslash CH_S(\Omega) \tag{3}$$

Theorem 4. Let $(\Omega,\ v)$ be the game defined in $\Omega\subset 2^N$, in which there exists a concatenation element$S$. If $Core(\Omega_S,\ v_S)\cap Weber(\Omega_S,\ v_S)=\emptyset$ or $Core(\Omega^S,\ v^S)\cap Weber(\Omega^S,\ v^S)=\emptyset$ then $Core(\Omega,\ v)\cap Weber(\Omega,\ v)=\emptyset$ holds true.

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A Hybrid PCA-NN-Kriging Framework for Geochemical Mapping: Distinguishing Natural from Anthropogenic Controls Using Multi-Element Data

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This study proposes a five-step hybrid framework integrating Principal Component Analysis, spatial diagnostics, neural networks, and kriging to predict soil element concentrations while distinguishing natural from anthropogenic controls. PCA decomposes multi-element data into scores and loadings, with reconstruction error analyzed spatially to determine whether kriging is needed. A neural network trained on terrain variables predicts PC1 and PC2, which following [1] are interpreted as local variability and areal background respectively, with the correlation between area-level PC2 means and PC1 variances serving as a diagnostic for natural control. The inverse PCA transform reconstructs element concentrations from predicted scores, enabling simultaneous prediction of any element in the original dataset. The framework is applied to three datasets: two neighboring natural watersheds on Olkhon Island (Khuzhir and Kharansy) and one potentially impacted gold mining territory (Buraevskaya) in the Irkutsk region.

Keywords: Geochemical mapping, PCA reconstruction error, kriging, neural networks, anthropogenic vs. geogenic sources, Olkhon Island, Jupyter notebooks.

Understanding and property relationship interpretation [1] of the spatial distribution of potentially toxic elements in soils is critical for environmental risk assessment, yet distinguishing between natural geogenic sources and anthropogenic inputs remains a fundamental methodological challenge. This study proposes a five-step hybrid framework that integrates Principal Component Analysis [2], spatial diagnostics [3], neural networks [4], and kriging [5] to predict element concentrations while providing interpretable diagnostics of the dominant controlling factors. The framework is designed to be fully reproducible, implemented in a series of Jupyter notebooks, and applicable to any region with geochemical samples and terrain data. We apply the framework to three distinct datasets: two neighboring watersheds on Olkhon Island in Lake Baikal — Khuzhir and Kharansy — representing natural background territories with minimal anthropogenic influence, and one mainland territory, Buraevskaya, which is a potentially gold mining area where anthropogenic impact is suspected but not yet confirmed by field studies.

The first step applies PCA to the multi-element geochemical data matrix X of dimension n samples by p elements, decomposing it as $X = T \cdot P^{\mathsf{T}} + E$, where T $(n \times k)$ are the scores representing projections onto the first k principal components, P $(p \times k)$ are the loadings defining the new coordinate axes, and E $(n \times p)$ is the reconstruction error representing information discarded during dimensionality reduction [2]. Following the theoretical framework in [1], the first two principal components receive substantive geographical interpretation. However, it is crucial to understand that PC1 and PC2 are mathematically orthogonal within a single dataset, meaning their correlation across individual samples is zero by construction. Analysis method in [1] operates at a different level of aggregation: across multiple geographical areas. For each area, we compute two statistics: (i) the mean PC2 value, interpreted as the stable areal background level that differentiates one geochemical province from another, and (ii) the variance (or standard deviation) of PC1, interpreted as the local intra-areal variability. According to the Gauss principle of least constraint [1], natural geosystems minimize their deviation from the environmental norm, which implies that across natural territories, areas with higher background levels should exhibit proportionally greater local variability. This relationship manifests as a linear correlation across areas: $r(\mathrm{PC2}_{\mathrm{mean}}, \mathrm{PC1}_{\mathrm{var}}) > 0.7$ for territories under natural geogenic control. The breakdown of this correlation — when a territory deviates from the line defined by natural areas — signals anthropogenic disturbance or unobserved local heterogeneity.

The second step analyzes the spatial structure of the reconstruction error E. For each element, we compute the empirical variogram of the corresponding column of E [3]. If the variogram is flat (pure nugget effect), the reconstruction error is spatially random, meaning that PCA has captured all geographically meaningful signal and no further spatial modeling is required. If the variogram exhibits increasing semivariance with distance, the reconstruction error contains spatial structure, indicating that PCA discarded geographically meaningful information. In this case, ordinary kriging of E is performed to recover the discarded signal, which typically represents either local geological heterogeneity (in natural areas) or anthropogenic input (in industrial areas). The need for kriging thus becomes a diagnostic output of the framework rather than an *a priori* assumption. This step operates independently for each dataset and does not require cross-area comparison.

The third step is the neural network prediction of principal components from observable terrain variables. A multilayer perceptron with hidden layers is trained to map terrain features — including spatial coordinates x and y, elevation, and slope — to the scores matrix T, i.e., $(\mathrm{PC1}, \mathrm{PC2}) = \mathrm{NN}(x, y, \mathrm{elev}, \mathrm{slope})$ [4]. This captures non-linear landscape controls on element dispersion that are not accessible to linear geostatistical models. The neural network operates independently of the kriging step; it predicts the retained signal $(T \cdot P^{\mathsf{T}})$, while kriging addresses the discarded signal (E). The NN is trained separately for each dataset, allowing it to learn the specific terrain-geochemistry relationships of each territory.

The fourth step reconstructs element concentrations from the predicted principal components. Given predicted scores T_{pred} from the neural network and the original loadings matrix P from the PCA decomposition, the inverse transform yields $X_{\mathrm{pred}} = T_{\mathrm{pred}} \cdot P^{\mathsf{T}}$, where X_{pred} is an $n_{\mathrm{pred}} \times p$ matrix of predicted concentrations for all p elements (As, Fe, Mn, Cu, Zn, Pb, etc.) at each new location [2]. This means that by training the neural network to predict only two or three principal components, the framework can simultaneously predict any element that was included in the original PCA, substantially reducing the dimensionality of the prediction problem while maintaining the ability to map individual elements of interest.

The fifth and final step integrates the kriging correction when indicated by the spatial analysis of E. If the reconstruction error exhibits significant spatial structure, the kriging predictions of E are added to the neural network predictions: $X_{\text{final}} = X_{\text{pred}} + E_{\text{kriged}}$. This ensures that no spatially structured signal is lost, whether it originates from local geological features missed by the terrain model or from anthropogenic sources not captured by the geochemical background.

We have prepared a complete set of Jupyter notebooks implementing this five-step framework and plan to apply them to three contrasting datasets in the Baikal region. The first two datasets are from Khuzhir and Kharansy, neighboring watersheds on Olkhon Island. Olkhon Island is a model clean territory with minimal anthropogenic influence, where all samples fall within a single geological class [6]. For these two natural territories, we expect high area-level correlation between PC2 means and PC1 variances, exceeding 0.7, indicating natural geogenic control of element distribution. We also expect the reconstruction error E to exhibit no significant spatial structure, confirming that PCA captures the full geographical signal and kriging adds no value.

The third dataset is from Buraevskaya, a potentially gold mining territory in the center of the Irkutsk region. Here we anticipate a fundamentally different pattern. The correlation between area-level PC2 means and PC1 variances is expected to fall below 0.7 when Buraevskaya is compared against the two Olkhon areas. This low correlation would suggest potential anthropogenic impact.

Our Jupyter notebooks, which implement all steps from PCA decomposition through spatial diagnostics to neural network prediction and element reconstruction, will be made publicly available and fully reproducible, enabling application of the framework to other regions including data-scarce environments typical of Global South countries such as Nigeria's mineralized belts and artisanal mining areas. The framework is transferable and can be extended to other heavy metals (Pb, Hg, Cd, Cu, Zn) for regional environmental monitoring.

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Analysis of Trajectories and Their Invariant Properties Formed by a Sequence of Two Types of Tractional-Linear Functions

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This thesis investigates discrete trajectories formed using two types of fractional-linear functions $f(x)$ and $g(x)$ (Möbius transformations). The primary objective of the research is to analyze the limit properties of the mappings $h_1(x) = f(g(x))$ and $h_2(x) = g(f(x))$ resulting from the composition of these functions.

Keywords: fractional-linear function (Möbius transformation), trajectory, composition, fixed points

If we are given a fractional-linear function $f(x) = \frac{x+a}{bx+c}$ $(b \neq 0, c \neq ab)$, then according to article [1], the trajectory $x_0, f(x_0), f^2(x_0), \ldots$ either converges at points $\forall x \in \mathbb{R} \backslash (\mathrm{Fix}(f) \cup \mathcal{P})$, or is periodic at points $\forall x \in \mathbb{R} \backslash \mathcal{P}$, or is dense in $\mathbb{R}$ at points $\forall x \in \mathbb{R} \backslash (\mathrm{Fix}(f) \cup \mathcal{P})$.

Now we need to study the convergence of the sequence:

$$x_{n+1} = \begin{cases} \frac{x_n+a_1}{b_1x_n+c_1}, n \text{ is even}, \\ \frac{x_n+a_2}{b_2x_n+c_2}, n \text{ is odd}. \end{cases}$$

Let $b_1 \neq 0,\ c_1 - a_1b_1 \neq 0,\ b_2 \neq 0,\ c_2 - a_2b_2 \neq 0,\ 1 + a_1b_2 \neq 0,\ 1 + a_2b_1 \neq 0.$

We introduce the following notations:

$$f(x) = \frac{x + a_1}{b_1x + c_1}, \quad g(x) = \frac{x + a_2}{b_2x + c_2},$$

$$h_1(x) = f(g(x)) = \frac{x + A_1}{B_1x + C_1}, \text{ where } A_1 = \frac{a_2 + a_1c_2}{1 + a_1b_2},\ B_1 = \frac{b_1 + b_2c_1}{1 + a_1b_2},\ C_1 = \frac{a_2b_1 + c_1c_2}{1 + a_1b_2},$$

$$h_2(x) = g(f(x)) = \frac{x + A_2}{B_2x + C_2}, \text{ where } A_2 = \frac{a_1 + a_2c_1}{1 + a_2b_1},\ B_2 = \frac{b_2 + b_1c_2}{1 + a_2b_1},\ C_2 = \frac{a_1b_2 + c_1c_2}{1 + a_2b_1}.$$

Let's write out the first few terms of the sequence

$$x_{n+1} \ : \ \{x_0, x_1, h_2(x_0), h_1(x_1), h_2^2(x_0), h_1^2(x_1), h_2^3(x_0), h_1^3(x_1), ...\}.$$

Therefore, the state of (1) can be fully determined through the trajectories $h_1 = \{x_1, h_1(x_1), h_1^2(x_1), ...\}$ and $h_2 = \{x_0, h_2(x_0), h_2^2(x_0), ...\}$.

The functions $h_1(x)$ and $h_2(x)$ can be constant, linear, or fractional-linear functions.

Case 1. Let $h_1(x)$ be a constant function. For the function $h_1(x) = \frac{x+A_1}{B_1x+C_1}$ to be constant, it must satisfy $C_1 = A_1B_1$. That is,

$$\frac{a_2b_1 + c_1c_2}{1 + a_1b_2} = \frac{a_2 + a_1c_2}{1 + a_1b_2} \cdot \frac{b_1 + b_2c_1}{1 + a_1b_2} \Rightarrow (c_1 - a_1b_1)(c_2 - a_2b_2) = 0$$

Lemma. The functions $h_1(x)$ and $h_2(x)$ cannot be constant functions.

If $h_1(x)$ were constant, then $h_2(x)$ would also have to be constant. However, since $c_1 - a_1b_1 \neq 0$ and $c_2 - a_2b_2 \neq 0$, this case is impossible.

Case 2. Let $h_1(x)$ be a linear function. For the function $h_1(x) = \frac{x+A_1}{B_1x+C_1}$ to be linear, it is necessary that $B_1 = 0$. That is, $\frac{b_1+b_2c_1}{1+a_1b_2} = 0$, which implies $b_1 + b_2c_1 = 0$.

Lemma. For the function $h_1(x)$ ($h_2(x)$) to be linear, it is necessary and sufficient that $b_1 + b_2c_1 = 0$ ($b_2 + b_1c_2 = 0$).

We can write $h_1(x) = \frac{1}{C_1}x + \frac{A_1}{C_1}$ and find its fixed points:

$$h_1(x) = x \quad \Rightarrow \quad \frac{1}{C_1}x + \frac{A_1}{C_1} = x \quad \Rightarrow$$

$$x_{01} = \frac{A_1}{C_1 - 1} = \frac{a_2 + a_1c_2}{a_2b_1 + c_1c_2 - 1 - a_1b_2} \quad (C_1 \neq 1).$$

Similarly, the fixed point of the function $h_2(x)$ will be:

$$x_{02} = \frac{a_1 + a_2c_1}{a_1b_2 + c_1c_2 - 1 - a_2b_1} \quad (C_2 \neq 1).$$

Let's write the general form of $h_1^n(x)$:

$$h_1^n(x) = \frac{1}{C_1^n}x + A_1\left(\frac{1}{C_1} + \frac{1}{C_1^2} + ... + \frac{1}{C_1^n}\right) = \frac{1}{C_1^n}x + A_1 \cdot \frac{1}{C_1} \cdot \frac{1 - \frac{1}{C_1^n}}{1 - \frac{1}{C_1}}.$$

Lemma. If $b_1 + b_2c_1 = 0$ and $\left|\frac{1}{C_1}\right| < 1$ (or $b_2 + b_1c_2 = 0$ and $\left|\frac{1}{C_2}\right| < 1$), then the trajectory h_1 (h_2) converges and $\lim_{n\to\infty} h_1^n(x) = x_{01}$ ($\lim_{n\to\infty} h_2^n(x) = x_{02}$).

Case 3. Let $h_1(x)$ and $h_2(x)$ be fractional-linear functions. We denote the sets of their "bad points" as follows:

$$\mathcal{P}_{h_1} = \left\{x \in \mathbb{R} : \exists n \in \mathbb{N} \cup \{0\}, \ \ h_1^n(x) = -\frac{C_1}{B_1}\right\},$$
$$\mathcal{P}_{h_2} = \left\{x \in \mathbb{R} : \exists n \in \mathbb{N} \cup \{0\}, \ \ h_2^n(x) = -\frac{C_2}{B_2}\right\}.$$

We find the fixed points of the function $h_1(x)$:

$$h_1(x) = x \quad \Rightarrow \quad \frac{x + A_1}{B_1x + C_1} = x \quad \Rightarrow \quad B_1x^2 - (1 - C_1)x - A_1 = 0 \quad \Rightarrow$$

$D_{h_1} = (1 - C_1)^2 + 4A_1B_1 = \frac{(a_2b_1+c_1c_2-1-a_1b_2)^2+4(b_1+b_2c_1)(a_2+a_1c_2)}{(1+a_1b_2)^2}$.
We introduce the following notations:

$$x_{h_{11}} = \frac{-(a_2b_1 + c_1c_2 - 1 - a_1b_2) + \sqrt{D_{h_1}}}{2(b_1 + b_2c_1)},$$

$$x_{h_{12}} = \frac{-(a_2b_1 + c_1c_2 - 1 - a_1b_2) - \sqrt{D_{h_1}}}{2(b_1 + b_2c_1)}.$$

It should be noted that the set of fixed points of the function $h_1(x)$ is as follows:

$$\mathrm{Fix}(h_1(x)) = \begin{cases} \{x_{h_{11}}, x_{h_{12}}\}, \text{ if } D_{h_1} > 0 \\ \{x_{h_{11}}\}, \quad \text{if } D_{h_1} = 0 \\ \emptyset, \quad \text{if } D_{h_1} < 0 \end{cases}.$$

According to Theorem 3.4 in article [1], for $\forall x \in \mathbb{R} \backslash (\mathrm{Fix}(h_1(x)) \cup \mathcal{P}_{h_1})$, the following holds:

$$\lim_{n\to\infty} h_1^n(x) = \begin{cases} x_{h_{12}}, \text{ if } D_{h_1} > 0 \text{ and } \left|\frac{\alpha_{h_1}}{\beta_{h_1}}\right| < 1 \\ x_{h_{11}}, \text{ if } D_{h_1} > 0 \text{ and } \left|\frac{\alpha_{h_1}}{\beta_{h_1}}\right| > 1 \\ x_{h_{11}}, \quad \text{if } D_{h_1} = 0 \end{cases},$$

where $\alpha_{h_1} = \frac{1+C_1+\sqrt{D_{h_1}}}{2}, \beta_{h_1} = \frac{1+C_1-\sqrt{D_{h_1}}}{2}$.
Similarly, for the function $h_2(x)$:

$$x_{h_{21}} = \frac{-(a_1b_2 + c_1c_2 - 1 - a_2b_1) + \sqrt{D_{h_2}}}{2(b_2 + b_1c_2)},$$

$$x_{h_{22}} = \frac{-(a_1b_2 + c_1c_2 - 1 - a_2b_1) - \sqrt{D_{h_2}}}{2(b_2 + b_1c_2)}.$$

$$\mathrm{Fix}(h_2(x)) = \begin{cases} \{x_{h_{21}}, x_{h_{22}}\}, \text{ if } D_{h_2} > 0 \\ \{x_{h_{21}}\}, \quad \text{if } D_{h_2} = 0 \\ \emptyset, \quad \text{if } D_{h_2} < 0 \end{cases}$$

$$D_{h_2} = (1 - C_2)^2 + 4A_2B_2 = \frac{(a_1b_2 + c_1c_2 - 1 - a_2b_1)^2 + 4(b_2 + b_1c_2)(a_1 + a_2c_1)}{(1 + a_2b_1)^2}.$$

For points $\forall x \in \mathbb{R} \backslash (\mathrm{Fix}(h_2(x)) \cup \mathcal{P}_{h_2})$:

$$\lim_{n\to\infty} h_2^n(x) = \begin{cases} x_{h_{22}}, \text{ if } D_{h_2} > 0 \text{ and } \left|\frac{\alpha_{h_2}}{\beta_{h_2}}\right| < 1 \\ x_{h_{21}}, \text{ if } D_{h_2} > 0 \text{ and } \left|\frac{\alpha_{h_2}}{\beta_{h_2}}\right| > 1 \\ x_{h_{21}}, \quad \text{if } D_{h_2} = 0 \end{cases},$$

where $\alpha_{h_2} = \frac{1+C_2+\sqrt{D_{h_2}}}{2}, \beta_{h_2} = \frac{1+C_2-\sqrt{D_{h_2}}}{2}$.
By simple calculations, it can be seen that the numerators in the expressions for D_{h_1} and D_{h_2} are equal, while the denominators are always positive. Thus, D_{h_1} and D_{h_2} either become zero simultaneously or always have the same sign. Therefore, the following lemma holds:

Lemma. The trajectories h_1 and h_2 are either both convergent or both divergent at the same time.

Theorem 1. For the sequence (1) to be convergent, it is necessary and sufficient that the trajectories h_1 and h_2 are convergent and their limits are equal.

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On Solving Some Operator Equations by Using Related Block Band Operators and Continued Fractions

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For some classes of operator equations we proposed an approach to their solution based upon the use of operator continued fractions and associated block band operators.

Keywords: operator equations, Band operators, Continued fractions

1 The main results

We start with the following equation in the unknown X

$$X^2 - zX + XU + V = O, \quad U, V, X \in L(H), \quad z \in \mathbb{C}, \tag{1}$$

such that H is a separable Hilbert space; V and X are invertible operators. It can be of interest in stability theory of differential equations and in the theory of operator bundles (factorization problem). We can associate with (1) the infinite block matrix

$$A = A(U,V) = \begin{pmatrix} U\,E\,O\,O\dots \\ V\,U\,E\,O\dots \\ O\,V\,U\,E\dots \\ \vdots\;\;\vdots\;\;\ddots\;\ddots\;\ddots \end{pmatrix}, \quad E,\, O - \text{identity and zero operators,}$$

and the block band operator A generated by A in the Hilbert space $l^2(H;[0,\infty)) := \{u = (u_i)_{i=0}^{\infty} \,|u_i \in H,\ \sum_{i=0}^{\infty}(u_i,u_i)_H < \infty\}$. Also, for $n \geq 1$ we consider the finite truncations of matrix A of size $n \times n$:

$$A^n = \begin{pmatrix} U\ E\ O \ldots O \\ V\ U\ E \ldots O \\ \ddots\ \ddots\ \ddots \\ O \ldots V\ U\ E \\ O \ldots O\ V\ U \end{pmatrix},$$

and the corresponding operators A^n acting in H^n.

Theorem 2 *The equation* (1) *has an exact solution*

$$X = zE - U - M(A,z)V$$

for $z \in D := \mathbb{C}\ \overline{W(A)}$, *where* $M(A,z)$ *is the Weyl (operator -) function of the operator* A *generated by matrix* $A(U,V)$ *and* $W(A)$ *is the numerical range of* A. *Its approximate solutions* X_n *are defined as follows:*

$$X_n = zE - U - M^n(z)V,$$

where $M^n(z)$ *are the Weyl functions of* A^n. *They converge to* X *locally uniformly in the domain* D *at a geometric rate.*

The proof is based on expansion of $M(A,z)$ into the operator continued $J-$ fraction [1]

$$J(U,V,z) = \frac{E}{zE-U} - \frac{V}{zE-U} - \frac{V}{zE-U} - \ldots$$

and its convergence to $M(A,z)$ on D at a geometric rate. We also consider two families of cubic and higher order operator equations:

$$X^m - zX^{m-1} + V = O; \qquad \text{and} \qquad X^m - zX + V = O, \quad m > 2,$$

and discuss how the vector continued fractions with operator coefficients similar to the ones studied in [2] can be utilized for their solving.

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Banach Type Fixed Point Results in Expanded M_b-Metric Spaces and Application to Nonlinear Integral Equations*

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This paper introduces a novel concept of expanded M_b-metric space as a generalization of M-metric space and expanded b-metric space. And we established Banach-type fixed point result in setting of expanded M_b-metric spaces. The obtained results are an improvement of some well-known fixed point results in M-metric spaces and M_b-metric spaces. Finally, we showed the existence and uniqueness of solutions for nonlinear integral equations as an application of our fixed point result.

Keywords: fixed point, expanded M_b-metric space, M_b-metric space, expanded b-metric space, nonlinear integral equation

1 Expanded M_b-metric space

Fixed point technology has gained increasing interest due to its wide application in many fields including nonlinear analysis, computer science, engineering, and economics, etc. One of the important research directions for extending Banach's contraction principle is the generalization of metric space. In 2014, Asadi et al. [1] first proposed another generalization of metric space called M-metric space in the fields of fixed point theory. Recently, several generalized concepts such as M_b-metric space, extended M_b-metric space, controlled M-metric type space and double controlled M-metric space have been introduced, and the related fixed point results have been established. (see [2,3,4,5])

We first introduce a new concept of expanded M_b-metric space which is generalization of M-metric space. We generalize M-metric space using the composition of a function as in the expanded b-metric space of [6].

Definition 1. Let $f \in \mathcal{F}$. Here $\mathcal{F}$ denotes the set of all onto mappings $f : [0, \infty) \to [0, \infty)$ such that $u \leq f(u)$, $\forall u \geq 0$ and $\acute{f}$ is increasing ($\acute{f}$ denotes derivative of f). A mapping $m_{Eb} : {}^2 \to [0, \infty)$ on $\neq \emptyset$ is called an expanded M_b-metric if the following conditions are satisfied:

(Emb-1) $\omega = \nu \Longleftrightarrow m_{Eb}(\omega, \omega) = m_{Eb}(\nu, \nu) = m_{Eb}(\omega, \nu)$;

(Emb-2) $m^{Eb}_{\omega\nu} \leq m_{Eb}(\omega, \nu)$;

(Emb-3) $m_{Eb}(\omega, \nu) = m_{Eb}(\nu, \omega)$;

(Emb-4) $m_{Eb}(\omega, \nu) - m^{Eb}_{\omega\nu} \leq f\left(m_{Eb}(\omega, \kappa) - m^{Eb}_{\omega\kappa}\right) + f\left(m_{Eb}(\kappa, \nu) - m^{Eb}_{\kappa\nu}\right)$, $\forall \omega, \nu, \kappa \in \Omega$,

where $m^{Eb}_{\omega\nu} := min\{m_{Eb}(\omega, \omega),\ m_{Eb}(\nu, \nu)\}$ and $M^{Eb}_{\omega\nu} := max\{m_{Eb}(\omega, \omega),\ m_{Eb}(\nu, \nu)\}$.

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The pair $(\Omega,\ m_{Eb})$ is called an expanded M_b-metric space.

Clearly, every M-metric space is an expanded M_b-metric space with $f(u) = u$ for all $u \geq 0$ and every expanded b-metric space is an expanded M_b-metric space. However, the converse is generally not true.

2 Fixed point result and application

We present the Banach-type fixed point theorem in the framework of expanded M_b-metric space.

Theorem 1. *Let* $(\Omega,\ m_{Eb})$ *be a complete expanded* M_b-metric space *with* $f\in\mathcal{F}$*, and let* $\mathcal{P}{:}\rightarrow$ *be a mapping satisfying*

$$m_{Eb}\left(\mathcal{P}\omega,\ \mathcal{P}\nu\right) \leq k \cdot m_{Eb}\left(\omega,\ \nu\right), \forall\omega,\ \nu\in\Omega,$$

where $k \in [0,\ 1)$*. Then* $\mathcal{P}$ *has a unique fixed point* $\omega^*\in\Omega$ *with* $m_{Eb}\left(\omega^*,\ \omega^*\right) = 0$.

In Theorem 1, setting $f(u) = u$ yields a Banach-type fixed point result in M-metric space (Theorem 3.1 in [1]), and setting $f(u) = su(s \geq 1)$ yields a Banach-type fixed point result in M_b-metric space (Theorem 2.1 in [2]).

We apply Theorem 1 to show the existence and uniqueness of solution for the following nonlinear integral equation

$$\omega(t) = \int_0^1 \mathcal{K}(t, s,\ \omega\left(s\right))ds, \forall t, s \in [0, 1]$$

where $\mathcal{K}{:}\,[0,1] \times [0,1] \times\mathbb{R}\rightarrow\mathbb{R}$ is given continuous function.

Let $\Omega{=}\ C(\,[0,1]\,,\ \mathbb{R})$. We define $m_{Eb} : \Omega^2\rightarrow[0,\ \infty)$ as

$$m_{Eb}\left(\omega,\ \nu\right) = sinh\left(max_{t\in[0,1]}\left(\frac{|\omega\left(t\right)| + |\nu\left(t\right)|}{2}\right)\right) \text{ for all } \omega,\ \nu\in\Omega.$$

We can check that $(\Omega,\ m_{Eb})$ is a complete expanded M_b-MS with $f(u) = sinh(2u)$.

Then, the above Fredholm integral equation has a unique solution in Ω under the following assumption:

$$sinh\left(\frac{|\mathcal{K}(t, s,\ \omega\left(s\right)| + |\mathcal{K}(t, s,\ \nu\left(s\right)|}{2}\right) \leq k \cdot sinh\left(\frac{|\omega\left(s\right)| + |\nu\left(s\right)|}{2}\right)$$

for all $\omega,\ \nu\in\Omega$ and $t, s \in [0, 1]$, where $k\in[0,\ 1)$.

Our future work will focus on the fixed-point results for different types of generalized contraction mappings in expanded M_b-metric space, and the deep and non-trivial applications.

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Supervised Functional Principal Component Regression for Estimation of Functional Response Model

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In this paper, we propose an estimation method of a functional response model with the functional principal components estimated in a supervised way which takes into account also the correlation with the response function.

Keywords: supervised, functional principal component analysis, functional response

1. Supervised FPCA

$$Y_i(t) = \alpha(t) + \int (t,s)X_i(s)ds + \varepsilon_i(t),\ \ i = 1,\, 2, ...,\, N \tag{1}$$

Let assume that $E(X(t))$=0 and $E(Y(t))$=0 in the following discussion. We can centralize $X(t)$ and $Y(t)$.

We estimate FPC $v_1(t)$, $v_2(t)$,...as follows. The estimate $\hat{v}_k(t)$ maximizes

$$Q(v) = \frac{\theta < v, Cv > +(1-\theta)E\,[< Y, v >]^2}{\|v\|^2}, \tag{2}$$

subject to $\|v\| = 1$, $< v, \hat{v}_j >= 0$, for every $j<k$ and $0\leq\theta\leq 1$. Here the norm $\|v\| = \sqrt{\|v\|^2} = \sqrt{< v, v >}$ and $<f,\, g>$ denotes the usual L^2 inner product.

2. Smooth Supervised FPCA

The FPCs obtained using (2) might need to be further smoothed or regularized. We define another type of norm as$\|f\|_\lambda = \sqrt{\|f\|^2 + \lambda\,\|D^2 f\|^2}$, in which$D^2 f = \int_T f''(t)dt$. The smooth estimate for the k-th supervised FPC is obtained by maximizing

$$Q(v) = \frac{\theta < v, Cv > +(1-\theta)\{E[< Y, v >]\}^2}{\|v\|_\lambda^2}, \tag{3}$$

subject to $\|v\|_\lambda = 1$, $< v, \hat{v}_j >= 0$, for every $j<k$ and $0\leq\theta\leq 1$.

3. Estimation of functional response model by supervised functional principal component regression

Let $\Phi(t)$=($\varphi_1(t)$, $\varphi_2(t)$,... $\varphi_M(t))^T$, we can rewrite $(X_1(t),\ X_2(t),...,X_n(t))^T$=$S\Phi(t)$. Similarly, let's rewrite $(Y_1(t),\ Y_2(t),..., Y_n(t))^T$=$R\Phi(t)$. In addition, we represent

$$\upsilon(t) = \sum_{m=1}^{M} \beta_m \phi_m(t) = {}^T\Phi(t), u(t) = \sum_{m=1}^{M} \gamma_m \phi_m(t) = {}^T\Phi(t)$$

=(β_1, β_2,..., $\beta_M)^T$ and =(γ_1, γ_2,..., $\gamma_M)^T$.

Then the empirical covariance function is

$$\hat{C}(s,t) = \frac{1}{n}\Phi(s)^T S^T S \Phi(t).$$

Therefore, we can see that

$$< \upsilon, C\upsilon >= \frac{1}{n}{}^T W S^T S W, \tag{4}$$

where **W** is an $M\times M$ matrix with elements w_{ij}=<$\varphi_i(t)$, $\varphi_j(t)$>.

For each $Y_i(t)$, <Y_i, υ> is written as <Y_i, υ>=${}^T W R_i^T$ where $\mathbf{R}_i$ is the i-th row of the coefficient matrix **R.**

$$E[< Y, \upsilon >]^2 = \frac{1}{n}\sum [< Y_i, \upsilon >]^2 = \frac{1}{n}\beta^T W R^T R W \beta, \tag{5}$$

The norm of $\upsilon(t)$ is given by

$$\|\upsilon\|_\lambda^2 = {}^T W + \lambda {}^T D = {}^T G, \tag{6}$$

where **D** denotes a $M\times M$ matrix with element d_{ij}=<D^2 $\varphi_i(t)$, D^2 $\varphi_j(t)$> and **G**=**W**+λ**D**.

Putting (4), (5) and (6) together, $Q(\upsilon)$ in (3) is given by

$$Q(\upsilon) = \frac{{}^T V}{{}^T G},$$

where $V = \frac{\theta}{n} W S^T S W + \frac{1-\theta}{n^2} W R^T 1_n 1_n^T R W$.

Let $= G^{1/2}$, maximizing $Q(\upsilon)$ is equivalent to maximizing $\delta^T(\mathbf{G}^{-1/2})^T\mathbf{V}\mathbf{G}^{-1/2}\delta$ subject to $\delta^T\delta$=1. Then δ_1,..., δ_p will be the leading p eigenvector of the matrix $(\mathbf{G}^{-1/2})^T\mathbf{V}\mathbf{G}^{-1/2}$.

Consequently, we can derive $\hat{}_j = (G^{1/2})^{-1}{}_j$. The corresponding smooth supervised FPC is $\hat{\upsilon}_j(t) = \hat{}_j^T\Phi(t)$ for j=1,...,p.

$\hat{u}_i(t)$ can be estimated by applying smooth unsupervised FPCA to $Y(t)$.

Similarly, if we let $= G^{1/2}$, we can derive $\hat{}_i = (G^{1/2})^{-1}{}_i$. The corresponding smooth FPC is $\hat{u}_i(t) = \hat{}_i^T\Phi(t)$ for j=1,..., q.

With the estimated leading p FPCs $\hat{\upsilon}_1(t), \hat{\upsilon}_2(t), ..., \hat{\upsilon}_p(t)$, we can estimate functional regression model of $Y(t)$ on $X(t)$.

$$Y(t) = \int \psi(t,s)X(s)ds + \varepsilon(t)$$

The slope function is given by

$$\hat{\psi}(t,s) = \sum_{k=1}^{q}\sum_{\ell=1}^{p} \frac{\hat{\sigma}_{\ell k}}{\hat{\lambda}_\ell}\hat{u}_k(t)\hat{\upsilon}_\ell(s),$$

where

$$\hat{\sigma}_{\ell k} = \frac{1}{N}\sum_{i=1}^{N} < X_i, \hat{v}_\ell >< Y_i, \hat{u}_k >, \hat{\lambda}_\ell = \frac{1}{N}\sum_{i=1}^{N} < X_i, \hat{v}_\ell >^2 .$$

Let $x(s)$ denotes $x(s) = \sum_{i=1}^{M} c_i \phi_i(s)$, then the estimate for response $y(t)$ is given by

$$\begin{aligned} \hat{y}_i(t) &= \int_T \sum_{k=1}^{q}\sum_{\ell=1}^{p} \frac{\hat{\sigma}_{\ell k}}{\hat{\lambda}_\ell} \sum_{m=1}^{M} \hat{\gamma}_{km}\phi_m(t) \sum_{m=1}^{M} \hat{\beta}_{\ell m}\phi_m(s) \sum_{i=1}^{M} c_i\phi_i(s)ds \\ &= \sum_{k=1}^{q}\sum_{\ell=1}^{p} \frac{\hat{\sigma}_{\ell k}}{\hat{\lambda}_\ell} \sum_{m=1}^{M} \hat{\gamma}_{km}\phi_m(t) \int_T \sum_{m=1}^{M} \hat{\beta}_{\ell m}\phi_m(s) \sum_{i=1}^{M} c_i\phi_i(s)ds = \sum_{k=1}^{q}\sum_{\ell=1}^{p} \frac{\hat{\sigma}_{\ell k}}{\hat{\lambda}_\ell}{}^T_\ell W C_k^T \Phi(t) \end{aligned} \tag{7}$$

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A Study on the Improvement of Control Iteration Algorithm for the Value Function Construction of High-Dimensional Minimum-Time Control Problem

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In this paper, we study the improvement of the control iteration algorithm for the value function construction of the minimum-time control problem. The control update process of the proposed control iteration algorithm involves a black-box optimization problem, which leads to a long computation time per iteration of the control iteration. Hence, we use gradient information to improve the performance of the control update process of the control iteration method and verify its effectiveness through numerical experiments.

Keywords: minimum-time control problem, control iterative algorithm, value function

We study an algorithm for the construction of a value function of a high-dimensional minimum-time control problem.

In [1], the improvement of the control update process of the control iteration is presented as a problem. In particular, the control update process is to solve a black-box optimization problem, which results in a long computation time per iteration of the control iteration. This is because the performance of the same algorithms for solving the black-box optimization problem is not high.

We investigate to update the control iteration method, overcoming this drawback of the control iteration method.

The computationally expensive part of this control iteration algorithm is Step 4.

In Step 4, zero-order optimization methods without using derivatives are applicable and thus take a long computation time.

$V(x)$ is known, the optimal control *a(x)* is

$$a\left(x\right) = \underset{a \in S_1}{\operatorname{argmin}} \left\{V(x) \cdot f(x,a)\right\}$$

We replace the control update step 4 by

$$a^{k+1}\left(x\right) = \underset{a \in S_1}{\operatorname{argmin}} \left\{V^{k+1}(x) \cdot f(x,a)\right\}$$

Updated control iteration algorithm.

Step 1. We discretize into discrete grids X, set Δt and set the initial value of the function V^0 at the grid points. The initial control at each grid point is set to a^0. Let be *k=0*.

Step 2. We fix $V^k = 0$, $x \in X \cap$

Step 3. We calculate

$$V^{k+1}\left(x\right) = e^{-\Delta t} I\left[V^k\right]\left(x + \Delta t f\left(x, a^k\left(x\right)\right)\right) + 1 - e^{-\Delta t}, x \in X \backslash$$

Step 4. According to the above gradient calculation procedure, we compute $V^{k+1}\left(x\right)$, $x \in X \backslash$.

Step 5. For $x \in X \backslash$, we calculate

$$a^{k+1}\left(x\right) = \underset{a \in S_1}{\operatorname{argmin}} \left\{V^{k+1}(x) \cdot f(x,a)\right\}$$

Step 6. If $\left\|V^{k+1} - V^k\right\| < \varepsilon$, stop, output V^k, or set to *k=k+1* go to step 3.

We will show the convergence of the updated control iteration algorithm. Convergence is generally proved for the continuous case, not for the discretized algorithm, as in algorithms above. The updated control iteration algorithm for the continuous case is as follows.

Step 1. Let be the computational domain, and let Δt be the time-division, the initial-value function V^0 set, and the initial control set a^0. Let be *k=0*.

Step 2. We fix $V^k = 0$, $x \in$.

Step 3. We calculate

$$V^{k+1}\left(x\right) = e^{-\Delta t} V^k\left(x + \int_0^{\Delta t} f\left(x, a^k\left(\cdot\right)\right) dt\right) + 1 - e^{-\Delta t}, x \in \backslash$$

Step 4. We calculate

$$a^{k+1}\left(x\right) = \underset{a \in S_1}{\operatorname{argmin}} \left\{V^{k+1}(x) \cdot f(x,a)\right\}$$

Step 5. If $\left\|V^{k+1} - V^k\right\| < \varepsilon$, stop, output V^k, or go to step 3.

In the continuous case, the control iteration algorithm converges.

[Lemma] If the initial value function is Lipschitz continuous, then at any step, is Lipschitz continuous.

[Theorem] For the algorithm of the continuous case, the conventional control iteration algorithm and the updated control iteration algorithm generate the same iterative sequence, so the updated control iteration algorithm converges to the optimal solution.

Conduct numerical experiments to verify the validity of the updated control iteration algorithm.

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On a Comparison Method for Studying the Existence of Solutions to Operator Equations in Vector Metric Spaces*

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We study the solvability of the equation $G(x,x)=\tilde{y}$, where $G : X \times X \to Y$, and X, Y are vector metric spaces (with distances taking values in some cones E_+, M_+ of a Banach space E and a linear space M respectively) by comparing it with a so-called "model" equation, $g(\varsigma,\varsigma)=0$, where a continuous map $g : E_+ \times E_+ \to M_+$ is orderly covering in the first argument and antitone in the second one.

Keywords: vector metric space, operator equation

Let M be a linear space and M_+ be an acute and convex cone in M. In M we define a "natural" order, that is $\nu \le \mu$ for $\nu, \mu \in M$ if $\mu - \nu \in M_+$; $\nu < \mu$ if $\nu \le \mu$ and $\nu \ne \mu$; $[\underline{\mu}, \overline{\mu}]_M := \{\nu \in M : \underline{\mu} \le \nu \le \overline{\mu}\}$. Given a nonempty set Y, a map $\mathcal{P}_Y^M : Y \times Y \to M_+$ satisfying for any $y, z, w \in Y$ the relations

$$\mathcal{P}_Y^M(y,z)=0 \;\Leftrightarrow\; y=z; \quad \mathcal{P}_Y^M(y,z)=\mathcal{P}_Y^M(z,y); \quad \mathcal{P}_Y^M(y,w) \le \mathcal{P}_Y^M(y,z)+\mathcal{P}_Y^M(z,w),$$

is called a *vector metric* or a v-*metric*, and a pair $(Y, \mathcal{P}_Y^M)$ a *vector metric* (v-*metric*) *space*. By $B_Y(y_0,\nu) := \{y \in Y : \mathcal{P}_Y^M(y,y_0) \le \nu\}$ denote a *ball of radius* $\nu \in M_+$ *centered at* $y_0 \in Y$.

For more on v-metric spaces see e.g., [1,2].

Now, consider two v-metric spaces $(X, \mathcal{P}_X^E)$ and $(Y, \mathcal{P}_Y^M)$, where E is a Banach space with closed regular convex and acute cone E_+, and M is a linear space with convex and acute cone M_+. We assume that the space $(X, \mathcal{P}_X^E)$ is complete (the convergence of a sequence $\{x_i\}$ to x in the space X means the convergence $\|\mathcal{P}_X^E(x_i,x)\|_E \to 0$, and the concepts of a closed set and a fundamental sequence are defined in a standard way).

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Let $G : X \times X \to Y$, $g : E_+ \times E_+ \to M$, and $\tilde{y} \in Y$. We consider the equation

$$G(x,x) = \tilde{y} \tag{1}$$

with respect to $x \in X$ and the corresponding "model" equation

$$g(\varsigma,\varsigma) = 0 \tag{2}$$

with the unknown $\varsigma \in E_+$. We discuss the conditions under which the solvability of the "model" equation (2) guarantees the solvability of equation (1).

Given a nonempty closed set $\mathfrak{B} \subset X$, we say that the map G is *closed with respect to the sets* $\mathfrak{B}$ *and* $\{\tilde{y}\} \subset Y$, if for any two sequences $\{x_i\}, \{x_i'\} \subset \mathfrak{B}$ such that $x = \lim_{i\to\infty} x_i = \lim_{i\to\infty} x_i'$ and $G(x_i, x_i') = \tilde{y}\ \forall i \in \mathbb{N}$, the limit x satisfies the equality $G(x,x) = \tilde{y}$.

Let $x_0 \in X$, $\overline{e} \in E_+$. We say that the map g *majorizes the map* G *on the ball* $\mathfrak{B} := B_X(x_0, \overline{e})$, if G, g, as the maps of the first argument, satisfy the relation

$$\forall e \in \mathfrak{I} := [0,\overline{e}]_E \ \ \forall x \in B_X(x_0,e) \ \ \forall \Delta \in [0, \overline{e}-e]_E$$
$$\mathcal{P}_Y^M(\tilde{y}, G(x,x)) \le g(e+\Delta, e) - g(e,e) \implies \exists\, u \in \mathfrak{B} \ \ G(u,x) = \tilde{y}, \ \ \mathcal{P}_X^E(u,x) \le \Delta,$$

and, as the maps of the second argument, the relation

$$\forall e \in \mathfrak{I} \ \ \forall \varsigma \in [0,e]_E \ \ \forall x \in B_X(x_0,e) \ \ \forall u \in B_X(x_0,\varsigma)$$
$$\mathcal{P}_X^E(u,x) \le e - \varsigma \implies \mathcal{P}_Y^M(G(x,u), G(x,x)) \le g(e,\varsigma) - g(e,e).$$

Next, suppose that the solutions set of equation (2) is nonempty and $\varsigma^* \in E_+$ is a minimal element in this set, and let the map g satisfy the following conditions:

(**g0**) $g(0,0) \le 0$;

(**g1**) g is *closed with respect to the sets* $\mathfrak{I}^* := [0,\varsigma^*]$ *and* $\{0\} \subset M$, i.e. for any two sequences $\{e_i\}, \{e_i'\} \subset \mathfrak{I}^*$ convergent to the same limit $e = \lim_{i\to\infty} e_i = \lim_{i\to\infty} e_i'$, and such that $g(e_i', e_i) = 0\ \forall i \in \mathbb{N}$, the limit e satisfies the equality $g(e,e) = 0$;

(**g2**) for any $\nu \in \mathfrak{I}^*$, the map $g(\cdot,\nu) : E_+ \to M$ *orderly covers the set* $\{0\} \subset M$ *on the interval* $\mathfrak{I}^*$, meaning that

$$\forall e \in \mathfrak{I}^* \ \ g(e,\nu) \le 0 \implies \exists e' \in \mathfrak{I}^* \ \ e' \ge e \text{ and } g(e',\nu) = 0;$$

(**g3**) for any $\nu \in \mathfrak{I}^*$, the map $g(\nu,\cdot) : E_+ \to M$ *is antitone on* $\mathfrak{I}^*$, i.e. for any $e, e' \in \mathfrak{I}^*$, the relation $e \le e'$ implies $g(\nu,e) \ge g(\nu,e')$.

Theorem. [3] Let the solutions set to equation (2) be nonempty and $\varsigma^* \in E_+$ be a minimal element in this set. Suppose that the map g satisfies conditions (**g0**)–(**g3**) and majorizes the map G on the ball $\mathfrak{B}^* := B_X(x_0,\varsigma^*)$, the map G is closed with respect to the sets $\mathfrak{B}^*$ and $\{\tilde{y}\}$, and the inequality $\mathcal{P}_Y^M(\tilde{y}, G(x_0,x_0)) \le -g(0,0)$ takes place. Then there exists a solution $x^* \in \mathfrak{B}^*$ to equation (1).

Remark. The result stated above is an analog of the Kantorovich fixed point theorem. The similar results for fixed points, for coincidence points, for solutions of general operator equations in normed and in metric spaces are obtained in [4,5,6,7].

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Variational Data Assimilation of Sea Surface Temperature for the Black Sea Hydrothermodynamics Model*

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In recent years, the study and numerical solution of variational data assimilation problems—which can be formulated as optimal control problems—have been widely developed in geophysics and oceanology. Data assimilation systems [1] are important tools that improve the quality of monitoring and forecasting the state of the environment in various water areas of the World Ocean.

The present work investigates the effect of the assimilation period duration of satellite data on the reproduction of sea surface temperature in a numerical model. The study uses data from the SNPP (VIIRS radiometer), Aqua, and Terra (MODIS spectroradiometer) satellites. Data from the SNPP satellite were selected as the assimilated information. The model calculation results were evaluated using sea surface temperature data from the Aqua and Terra satellites. These data were provided by the Collective Use Center 'IKI Monitoring' service [2]. The proposed data assimilation algorithm [3] allows calculations to be performed even when satellite data do not cover the entire water area under study. Unlike most variational data assimilation problems, in this work the control variable is the heat flux at the air–sea boundary, which is included in the cost functional. This makes it possible to use assimilation at each time step of the model, allowing calculations to be performed with a slight time delay relative to the main model run. The results of numerical experiments are presented for the Black and Azov Seas.

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Keywords: variational assimilation, satellite data, sea surface temperature, ocean hydrothermodynamics model

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Construction of Solutions to the Analogues Non-stationary of the Schrödinger Equations Corresponding to the Isomonodromic Hamiltonian System $H^{3+\frac{3}{2}}$

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A pair of joint solutions of the analogues non-stationary of the Schrödinger equations is constructed defined by the Hamiltonians $H^{3+1.5}_{s_k}(s_1, s_2, q_1, q_2, p_1, p_2)(k = 1, 2)$ of the Hamiltonian system from the Kimura-Kawamuko list. These analogues non-stationary of the Schrödinger equations are linear evolution equations with times s_1 and s_2, each of which depends on two spatial variables.

Keywords: Hamiltonian systems, Schrödinger equations, Painlevé equations, method of isomonodromic deformations

1 The main results

Along with the six classical Painlevé ODEs, in recent decades, there has been an increasing interest in higher-order nonlinear ODEs that are also integrated by the IDM method. Today, in particular, a finite list of joint pairs of Hamiltonian ODE systems is known.

$$(q_j)'_{s_k} = (H_{s_k})'_{p_j}, \quad (p_j)'_{s_k} = -(H_{s_k})'_{p_j} \quad (k = 1, 2) \quad (j = 1, 2)$$

with Hamiltonians $H_{s_k}(s_1, s_2, q_1, q_2, p_1, p_2)$, each of which is a compatibility condition two linear systems of ODEs of the form

$$V'_{s_k} = L_{s_k} V, \quad V'_\eta = AV,$$

where the square matrices L_{s_k} and A are of the same dimension and are rational in the variable η. This list is given in Kimura's paper, see [1]. Later, Kawamuko added to this list, see [2]. In this work, we will construct solutions to the following equations:

$$k\Psi'_{\tau_1} = H_1^{3+\frac{3}{2}}(\tau_1, \tau_2, x_1, x_2, -k\frac{\partial}{\partial x_1}, -k\frac{\partial}{\partial x_2})\Psi,$$

$$k\Psi'_{\tau_2} = H_2^{3+\frac{3}{2}}(\tau_1, \tau_2, x_1, x_2, -k\frac{\partial}{\partial x_1}, -k\frac{\partial}{\partial x_2})\Psi.$$

These equations are the analogues of the time-dependent Schrödinger equations. The author was only able to construct them for the case $k = 1$. For the case $k = i\hbar$ it was not possible. If it had been possible, then these would not be analogues, but the actual Schrödinger equations.

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Variational Problems of Nonlinear Elasticity on Carnot Groups*

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One approach to finding the configuration into which a hyperelastic solid Ω transforms as a result of the action of known external forces on it is to find the mapping $\varphi : \Omega \to \mathbb{R}^n$ that provides the minimum energy functional

$$I(\varphi) = \int_{\Omega} W(x, D\varphi(x))\, dx.$$

In the last century, J. Ball found mathematical conditions corresponding to real materials, under which it is possible to obtain a theorem on the existence of a minimum of the functional I in a certain class of continuous maps with generalized derivatives.

The paper [1] presents the application of modern quasi-conformal analysis methods to this problem. Based on them, in the class of maps with integrable distortion, the existence of an extreme map that is one-to-one is established. In [2], this approach is developed on Carnot groups, which have a significantly more complex geometry compared to the Euclidean space.

Keywords: nonlinear elasticity, quasi-conformal analysis, calculus of variations

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Nonlinear Analysis of Processes in Magnetohydrodynamic Systems

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Large-scale nonlinear oscillations of an electrically conducting ideal fluid of variable depth are considered, taking into account magnetic, buoyancy (Archimedean), and Coriolis forces, as well as magnetic field diffusion. The hydromagnetic pressure is approximated as a linear function depending on the layer depth. As a result of modeling, the investigated dynamical process is formulated as a nonlinear boundary value problem for the horizontal components of velocity and magnetic field, as well as for a function describing the free surface of the layer. The governing equations are derived using scale analysis of quasi-geostrophic motions. Assuming that the Rossby numbers, which characterize the ratio of local and advective accelerations to the Coriolis acceleration, are of the same order, the problem is reduced to a system of three nonlinear equations for the hydromagnetic pressure and two functions describing the magnetic field. In the case of a horizontally unbounded rotating electrically conducting fluid, and under the assumption of an approximately constant slope of the solid boundary confining the layer over distances of the order of the wavelength, an exact solution of the corresponding nonlinear system is obtained.

Keywords: mathematical modeling, magnetohydrodynamics, nonlinear waves, quasi-geostrophic approximation

1 The main results

In the presented study, analytical solutions are constructed for the problem of large-scale motions taking into account the quasi-geostrophic properties of an electrically conducting rotating fluid. A rectangular Cartesian coordinate system is used, and the body force is defined as a vector perpendicular to the undisturbed surface and directed along the vertical axis [1]. A rotating layer of an electrically conducting ideal incompressible fluid bounded by a known solid impermeable surface and an unknown free surface is considered. The system of equations describing fluid motion in magnetohydrodynamics includes the equations of motion and continuity, as well as the induction equation and the solenoidal condition for the magnetic field.

The system of partial differential equations describing the physical phenomena under consideration is nonlinear, non-autonomous, and high-dimensional. The construction of analytical, in particular exact, solutions is possible only for specific particular cases. In the general case, the original system must be approximated by a simpler system that adequately describes the properties of its solutions.

At the next stage of the study, a reduction method is applied, resulting in a single scalar partial differential equation. Its qualitative analysis makes it possible to draw conclusions about the influence of rotation and nonlinear magnetic diffusion effects on the investigated wave process [2].

The presented results can be applied in astrophysics and geophysics, in particular in the study of processes occurring in the Earth's liquid core and in stellar interiors [3]. The obtained solutions represent a simplified model of the Earth's liquid core in the form of a horizontally unbounded fluid layer, taking into account the topography of the mantle and the solid inner core. These studies may also be useful for analyzing the self-excitation process of magnetohydrodynamic dynamos in large volumes of liquid metal and technical devices. The use of strong magnetic fields is of particular importance in thermonuclear fusion, plasma physics, solid-state physics, and nuclear physics, where strong fields serve as essential and sometimes the only research tool. Magnetic measurements provide reliable diagnostics of various operating conditions of reactors, including malfunctions of pumping systems.

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Modeling the Cyclic Activity of Environment-Destroying Populations in the Taimyr Region

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Populations of wide variety of animal species exhibit pronounced fluctuations in abundance, including non-spontaneous ones. Distinct oscillations with a periodic component are observed. Harmonic cycles around equilibrium are rare phenomenon in biological reality. Population fluctuations occur with very different patterns and are caused by various factors. Invasive insects often exhibit repeating series

of decaying peaks. Many populations that damage their environment exhibit an abrupt break in the cycle after high peak. Critical declines occur due to trampling of pastures in the Taimyr region. Models are needed to describe abrupt changes in cyclic and depression patterns. Threshold trigger values must be included in equations for oscillatory modes. This article proposes a method for modeling changes in oscillatory patterns based on hybrid overdetermined equations with delay.

Keywords: hybrid equations with delay, depletion of populations in the Taimyr region, methods for organizing re-definable computational structures

This article explores computational analysis for the non-linear dynamics of complex biophysical processes that can exist in fundamentally different modes of existence. For biosystems, such modes can include chaotic pulses, oscillatory changes, cyclical recurring processes, or deep depression. An invasive population, after the peak of an outbreak and depletion of the environment, may alternatively exist in a refugium mode. These modes of biosystem change and phenomena are not described by a single general model, so the author consistently develops a hybrid modeling method for non-linear biophysics and threshold ecodynamics.

An adequate model of nonlinear biophysical processes requires justification for the shifting behavior patterns. The methodology challenge includes the biological justification for the choice of formalization. Particular difficulties arise when studying oscillatory regimes, as in reality, fluctuations are not stable but transient and collapse when a certain level of impact on environmental resources is exceeded. In real biophysical processes, oscillation patterns are extremely diverse and are not described by the known cycles of predator-prey models. This is relevant for mammal populations on the Arctic Taimyr Peninsula, which experience fluctuations with deep minima. Fluctuations in aggressive populations are more often sawtooth-like series. Reality bears little resemblance to the harmonic cycles of classical models of predator-prey interactions. With the introduction of delay and increased dimensionality, the cyclical behavior of the numerical solution of many popular models gives way to aperiodic behavior, and dynamic chaos arises. Including a third type of competitor in the model will complicate behavior with the potential for chaos, but in reality, dense food webs are more stable. It is shown that the initially correct behavior of models in scenarios suddenly becomes excessively complex when compared with known situations. For none of the studied problems did dynamic chaos help describe the situation in the model, but instead became a problem. The properties associated with chaos are excessive for predictive models of real biosystems, and the conclusions are contradictory. Aperiodic regimes hinder understanding the essence of evolution. The behavior of models based on systems of equations with delays becomes confusing and excessively sensitive to any changes in parameters and initial data. Significant uncertainty in the behavior of the solution cannot correspond to the basic principles of stabilizing evolution and the concept of ecological homeostasis and the collapse of Taimyr bioresources. The article discusses interpretability methods and criteria for essential distinctions for various types of irregular behavior in epidemic and population models. The niche capacity K in population growth models $\dot{N} = rf(N^k(t-\tau); K)$ present a theoretical problem. It is not clear how we can calculate the volume of the limiting niche for invaded species with greatly increased r? Analyzing population dynamics at low abundance presents both a theoretical challenge and a difficulty for the management of nature. Degradation and even eventual extinction of populations is unfortunately becoming more and more common nowadays. An important question is what population size is needed to guarantee its existence under environmental perturbation? The necessity of having a guaranteed minimum abundance of individuals $\hat{L} > L$ to maintain the stable existence of a population has been discussed for a long time. If the population is cyclic, is there bottom bounded $\min N_*(t; r\tau)$? The hypothesis of a population L minimum is reflected in the model, where L turns out to be unstable equilibrium $\forall N(t) < L, \lim_{t\to\infty} N(t) = 0$:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \times (N - L). \tag{1}$$

In the equation L threshold has effect even when the population is far from it $N(t) \gg L$. The author proposes to aggregate the ideas of the two models and the Θ-logistic (where $0 < \Theta < 1$ is justified) in the new, flexibly adjustable modification. Observations show that the success in offspring survival with an increase in $N(t)$ in large colonies of social insects, far exceeds the negative impact of intraspecific competition. Let's formalize the new model:

$$\frac{dN}{dt} = rN\left(1 - \left(\frac{N}{K}\right)^{\Theta}\right) \sqrt[3]{(N - L)}, \Theta \in [1/2, 3/4]. \tag{2}$$

The new model named $\Theta-$Bazykin equation is designed for invasive insect species that form large colonies. A small group size becomes dangerous for them, which is also true for activist communities on social networks during information attacks.

An optimal size of a group for reproduction is extremely significant. A decrease in abundance as $N(t) = K \to L$ does not always lead to extinction. Several relict species are known that have been in small but stable state for a long time. The scenario formation of local refuge for previously numerous dominant species in the evolution theory is one of the interesting rare variants of population processes. It is quite difficult to estimate the minimum group L for a specific species without experimental studies. On the other hand, some dangerous and rapid invasions began with single penetrating specimens. An alternative version of the minimum group effect model without explicit use of the L threshold is:

$$\frac{dN}{dt} = r_1 \frac{\gamma N^2}{\gamma + \sigma N} - \varsigma N - \delta N^2. \tag{3}$$

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On Observability of Differential-Algebraic Systems with Hysteresis Modeled by a Sweeping Process*

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In this paper, we investigate some class of differential-algebraic equations (DAEs) incorporating nonlinearities of hysteresis. For such a DAE sufficient conditions of observability are proved.

Keywords: differential-algebraic equations, hysteresis, sweeping process, solvability, observability

1 The main results

Consider the system of ordinary differential equations (ODE) paired with a sweeping process [1,2] of hysteresis type (modeled by the play-operator [3,4,5]):

$$A(t)\dot{x}(t) = B(t)x(t) + U(t)y(t), \quad x(t_0) = x_0, \tag{1}$$

$$-\dot{z}(t) \in \mathcal{N}_{Q(t)}(z(t)),\ z(t) = \dot{y}(t),\ y(t_0) = y_0,\ z(t_0) = z_0 \in Q(t_0), \tag{2}$$

$$h(t) = H(t)x(t). \tag{3}$$

Here $A(t), B(t), U(t) \in \mathbb{R}^{(n\times n)}, H(t) \in \mathbb{R}^{(m\times n)}$ are given matrices, wherein $\det A(t) \equiv 0$, $t \in T$, $T = [t_0, t_1]$ is a given time interval; $x(t), y(t) \in \mathbb{R}^n$ are unknown functions; $h(t)$ is an output function; $Q(t) = x(t) - Z$ is a “moving set”, where Z is a given closed convex subset of $\mathbb{R}^n$, and $\mathcal{N}_Q(q)$ denotes the normal cone to a closed convex set Q at a point q.

We define a solution to (1)–(3) as a triple (x, y, z) of absolutely continuous functions (belonging to $W^{1,1}(T)$) that satisfy the system for almost all $t \in T$.

Our study relies on a structural decomposition of the DAE into differential and algebraic parts. This decomposition preserves the solution set, and the transformation operator is left-invertible. The construction is explicit, avoids coordinate changes, and automatically resolves the consistency of initial conditions.

Consider time-varying ODE system not resolved with respect to the derivative

$$A(t)x'(t) = B(t)x(t) + U(t)f(t), \quad t \in T \subseteq \mathbb{R}, \tag{4}$$

where $A(t), B(t), U(t) \in \mathbb{R}^{(n\times n)}$ are given matrices, wherein $\det A(t) \equiv 0$, $f(t) \in \mathbb{R}^n$ is a continuous function on T; $x(t) \in \mathbb{R}^n$ is unknown function of a state. Such systems are called differential-algebraic equations (DAE).

Lemma 1. [6] *Let $A(t), B(t) \in \mathbb{C}^{r+1}(T), U(t) \in \mathbb{C}^r(T)$, there is a resolving minor in matrix $\mathcal{D}_r(t)$ and* rank $\mathcal{D}_r(t) =$ rank $\mathcal{D}_{r-1}(t) + n \quad \forall t \in T$. *Then there exists a linear differential operator*

$$\mathcal{R} = R_0(t) + R_1(t)\frac{d}{dt} + ... + R_r(t)\left(\frac{d}{dt}\right)^r$$

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with continuous on T coefficients $R_j(t)$ $(j = \overline{0, r})$, which reduces system (4) into the form

$$\dot{x}_1(t) = J_1(t)x_1(t) + (L_0(t)\ L_1(t)\ \dots\ L_r(t))\ \text{column}(f(t), \dot{f}(t), \dots, f^{(r)}(t)),$$

$$x_2(t) = J_2(t)x_1(t) + (G_0(t)\ G_1(t)\ \dots\ G_r(t))\ \text{column}(f(t), \dot{f}(t), \dots, f^{(r)}(t)),$$

where $(x_1(t), x_2(t)) = S^{-1}x(t)$, $x_1(t) \in \mathbb{R}^{n-d}, x_2(t) \in \mathbb{R}^d$, r is unsolvability index for DAE (4). Matrices $\mathcal{D}_r(t), J_1(t), J_2(t)$, $L_i(t), G_i(t)$ $(i = \overline{1, r})$ are determined by formulas cited in [6].

Theorem 1. *Let all the assumptions of Lemma 1 be satisfied for $r = 1$. Define the observability matrix $O(t)$ as a block column consisting of the matrices*

$$\mathcal{O}(t) = \text{column}(O_0(t)\ \ O_1(t)\ \ \dots\ O_{n-d-1}(t)),$$

$$O_0(t) = H(t),\ \ O_i(t) = O_{i-1}(t)J_1(t) + O'_{i-1}(t),\ \ i = \overline{1, n-d-1}.$$

Denote as:

1) rank $(G_0(t)\ G_1(t)) = d\ \ \forall t \in T$;

2) $\exists \sigma \in T :$ rank $\mathcal{O}(\sigma) = n - d$;

3) rank $\mathcal{O}(t) = n - d$ *for almost all* $t \in T$.

We introduce the following notions of observability for system (1)–(3):

- *R-observability holds if condition 2) is met.*
- *Complete observability holds if conditions 1) and 2) are met.*
- *Differential observability holds if conditions 1) and 3) are met.*

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On Feedback Control Systems Governed by Functional Inclusions with Causal Operators and a Sweeping Process*

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Controllability conditions are found for systems governed by functional inclusions with causal multioperators in a Banach space and feedback in the form of a sweeping process in a Hilbert space. Topological methods of nonlinear analysis for multivalued condensing maps are used to prove the general controllability principle. Particular cases of the problem under study are considered for first-order and fractional-order differential inclusions.

Keywords: functional inclusion, feedback, sweeping process, causal operator, measure of noncompactness, multivalued map, condensing multioperator

1 The main results

Let E be a Banach space and H a Hilbert space. Consider a feedback control system described by the following functional inclusion and sweeping process:

$$x(t) \in \mathcal{G}(t)x_0 + \mathcal{S} \circ \mathcal{Q}(x, y)(t) + \mathcal{S} \circ Bu(t), \ t \in [0, a], \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$- y'(t) \in N_{D(t)}(y(t)) + g(t, x(t), y(t)) + ly(t), \ t \in [0, a], \tag{3}$$

$$y(0) = y_0 \in D(0), \tag{4}$$

$$x(a) = x_1, \tag{5}$$

where $\mathcal{S} : L^p([0, a]; E) \to C([0, a]; E)$ is a causal operator, $\mathcal{Q} : C([0, a]; E) \times C([0, a]; H) \to Cv(L^p([0, a]; E))$ is a causal multioperator with closed convex values and the operator $\mathcal{G}$ has the form

$$\mathcal{G}(t) = \int_0^\infty \xi_q(\theta)T(t^q\theta)d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-1/q}), \quad \Psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq + 1)}{n!}\sin(n\pi q), \theta \in \mathbb{R}^+,$$

where $T(\cdot)$ is a C_0-semigroup of operators. Note that if $q = 1$, then $\mathcal{G}(\cdot) = T(\cdot)$. The external control function $u(\cdot)$ belongs to the space $L^\infty([0, a]; U)$, where U is a Banach control space and $B : U \to E$ is a bounded linear operator. Further, $D : [0, a] \to 2^H$ is a multivalued map with closed convex values. For the internal control y, $N_{D(t)}(y)$ denotes the normal cone defined by the closed convex set $D(t) \subset H$:

$$N_{D(t)}(y) = \begin{cases} \{\xi \in H : \langle \xi, c - y\rangle \le 0 \text{ for all } c \in D(t)\}, & \text{if } y \in D(t), \\ \emptyset, & \text{if } y \notin D(t). \end{cases}$$

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and $g : [0, a] \times E \times H \to H$ is a nonlinear map, $x_0, x_1 \in E,\ y_0 \in H$ are given, and $l > 0$.

The controllability problem is formulated as follows: for given x_0, x_1, the existence of solutions $x \in C([0, a]; E), y \in C([0, a]; H)$ of the system (1)-(4) and the control function $u \in L^\infty([0, a]; U)$ such that the condition (5) is satisfied.

The conditions under which a solution to the problem exists have been found.

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On Structural Stability of 3-diffeomorphisms with the Smale Solenoid Attractor-Repeller Dynamics*

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We consider 3-diffeomorphisms having source-sink dynamics, where Smale solenoids play the role of source and sink (NSSS-diffeomorphisms). It is known that all such diffeomorphisms exist only on lens spaces. At the same time, all the examples known to date are not structurally stable. In particular, each NSSS-diffeomorphism on the 3-sphere is associated with an exchangeable braid. In [6] for every $n > 2$ were constructed exchangeable braid with strand number n such that realized by every such a braid NSSS diffeomorphism on S^3 is not structurally stable. The braid of the classical Smale example has strand number 2 and it is exchangeable. However, the technique in [6] did not allow to prove that the corresponding NSSS-diffeomorphism on S^3 is not structurally stable. In this work, we fill the gap.

Keywords: smale solenoid, structural stability

1 The main results

Solenoids were first defined by L. Vietoris in 1927 [5] and were introduced into dynamics by S. Smale [4] for the study of hyperbolic basic sets. *Local geometry model of Smale solenoid* has the following description.

Let $V = \mathbb{D}^2 \times \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \times \{z = e^{i\theta} : 0 \leq \theta < 2\pi\}$ and $\pi : V \to \mathbb{S}^1$ be the natural projection. Let $\varphi(z) = z^n : \mathbb{S}^1 \to \mathbb{S}^1$ for some $n \in \mathbb{N}$, $n \geq 2$ and $\Phi : V \to V$ be an embedding map such that

(a) $\pi\Phi = \varphi\pi$;
(b) Φ is a $\mathbb{D}^2$-level-preserving, linear in θ embedding and the radius of $\Phi(\mathbb{D}^2, z)$ is $\frac{1}{n^2}$.

We call

$$S = \bigcap_{j=1}^{\infty} \Phi^j(V)$$

a Smale solenoid, which is a hyperbolic attractor of Φ. Every Smale solenoid attractor is associated to a *solenoid braid* σ_S whose closure $\hat{\sigma}_S$ is the core of $\Phi(V)$ in V. This allows us to define the *strand number of a Smale solenoid attractor* S as the number of strands of σ_S.

Let M be a closed orientable differential 3-manifold and $f : M \to M$ be an Ω-stable diffeomorphism (equivalently the chain recurrent set Ω_f of f is hyperbolic). One says that *f has a Smale solenoid attractor* Λ_a as a chain component if there is a neighborhood V_a of Λ_a and a homeomorphism $h : V \to V_a$ such that $h\Phi|_V = fh|_N$. Λ_r is *a Smale solenoid*

* The work is an output of a research project implemented as part of the Basic Research Program at the National Research University Higher School of Economics.

repeller if is a a Smale solenoid attractor for f^{-1}. A ***north-south Smale solenoid diffeomorphism (NSSS-diffeomorphism)*** is a diffeomorphism $f : M \to M$ with

$$\Omega_f = \Lambda_a \sqcup \Lambda_r.$$

For NSSS-diffeomorphisms, we still can define ***strand numbers*** by using the strand numbers of Smale solenoid attractors. An NSSS diffeomorphism f is called ***perfect*** if $M = V_a \cup V_r$ with $\partial V_a = \partial V_r$.

It was Bothe [1] who first realized some Smale solenoid attractors in some closed 3-manifolds. Jiang, Ni and Wang [2] showed that a closed 3-manifold M admits an NSSS-diffeomorphism if and only if M is homeomorphic to a lens space $L_{p,q}$, $p \neq 0$. Notice that one side of proof in [2] is constructive. They also showed that perfect NSSS-diffeomorphisms which they constructed are not structurally stable. It follows from [6] that no perfect NSSS-diffeomorphism is structural stable at all.

It is proved in [3] that a braid can be realized in an NSSS diffeomorphism on $\mathbb{S}^3$ if and only if it is an exchangeable braid.

Recall that ***an exchangeable braided link*** is a two-component link $L = K_1 \sqcup K_2$ in $\mathbb{S}^3$ so that each component is braided relative to the other one, that is K_1 is a closed braid $\hat{\sigma}_1$ in the solid torus $\mathbb{S}^3\ \ K_2$ and K_2 is a closed braid $\hat{\sigma}_2$ in the solid torus $\mathbb{S}^3\ \ K_1$. A braid σ is called ***an exchangeable*** if there exists an exchangeable braided link L so that the closure of σ is a component of L.

In [6] for every $n \geq 3$ were considered a braid σ with strand number n and a braid word $\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2} \ldots \sigma_{n-1}^{\varepsilon_{n-1}}$, where $\varepsilon_i \in \{-1, 1\}$ and there exist $i_1, i_2 \in \{1, \ldots, n-1\}$ such that $\varepsilon_{i_1}\varepsilon_{i_2} = -1$. It was established that every such a braid is exchangeable and any realized by it NSSS diffeomorphism on S^3 is not structurally stable.

The braid of the classical Smale example has strand number 2 and one-letter word braid. The technique in [6] did not allow to check the structural stability of the corresponding NSSS-diffeomorphism on S^3, which was the reason for writing this paper.

The main result of this paper to a prove the following fact.

Theorem 1. *Every NSSS-diffeomorphism on $\mathbb{S}^3$ with the strand number 2 is not structurally stable.*

The research is an output of a research project implemented as part of the Basic Research Program at the National Research University Higher School of Economics.

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Expenditure Optimized Control of Small Motions of Mechanical Systems

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The optimal controlled motion of a mechanical system, that is determined by the system ODE and piecewise constant control components, is considered. The number of control switching points of control steps are considered as preset. The optimized functional is equal to the total area of all steps of all control components ("Expenditure criteria"). Admissible controls are those that turn to zero (at a non predetermined time moment) the previously chosen frequency components of the solution. An algorithm for the finding of control switching points and the heights of control steps, based on the necessary minimum conditions for Expenditure criteria, is proposed.

Keywords: controlled motion of a mechanical system, control with expenditure criteria, bang-bang control

1 The main results

A method is proposed for constructing expenditure-optimal control of small motions of mechanical systems near an equilibrium position. Such motions may include undamped oscillations at various frequencies, the damping of which is necessary for the system to function correctly. From both a mathematical and an engineering standpoint, the most suitable approach for implementing such damping is a piecewise constant control action. Expenditure optimization implies minimizing the resources required for control and is modeled by a non-smooth functional expressed as the integral of the sum of the absolute values of the control vector components over the time interval from zero to the control cutoff time. A control is considered admissible if it brings oscillations of the selected frequency to zero at the cutoff time.

In studies [1,2,3,4,5], various analytical and numerical algorithms for constructing expenditure optimal control were proposed, both for specific problems and in a general formulation. In those works, control actions were taken from the classes of piecewise constant and piecewise polynomial functions, and only the switching times were used as optimization parameters. The present paper proposes an algorithm in which optimization is performed not only over the switching times but also over the step heights of the control. This expands the possibilities for expenditure minimization, allowing the selection of not only the best switching times but also the optimal control magnitude.

Thus, the following problem formulation is considered: for a given number of steps of an admissible piecewise constant control, find the switching points (including the control cutoff point) and the step heights of this control that satisfy the necessary extremum conditions for the expenditure functional.

The proposed analytical algorithm allows the computation of the optimal control to be automated.

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On the Asymptotic Stability of Systems of Differential Equations with Almost Periodic Coefficients*

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This paper examines the asymptotic stability of the zero solution of a differential equation with almost periodic coefficients in the linear terms, in terms of the solvability of the Lyapunov differential equation.

Keywords: almost periodic functions, Lyapunov matrix equations, Kreĭn theorem

1 The main results

We consider the problem of asymptotic stability of solutions to the following system of differential equations:

$$\dot{y}(t) = A(t)y(t), \tag{1}$$

where $A(t)$ is a uniformly almost periodic matrix function. For this class of functions, the following mean value theorem is well-known (see, e.g., [1]).

Theorem 1. *If $A(t)$ is an almost periodic matrix function, then there exists a matrix A_0 such that*

$$A_0 = \lim_{T\to\infty} \frac{1}{T} \int_x^{x+T} A(s)\, ds$$

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uniformly with respect to x.
We denote the mean value for an almost periodic matrix function $A(t)$ as follows:

$$\lim_{T\to\infty} \frac{1}{T} \int_x^{x+T} A(s)\,ds = M\{A(t)\} = M_t\{A\}.$$

For the solutions of system (1) under small perturbations of the matrix $A(t)$, we can obtain an analogue of the Krein estimate (see, e.g., [2]).

Theorem 2. *Let $M\{A(t)\} = A_0$. Suppose that the spectrum of the matrix A_0 belongs to the left half-plane $C\_$ and the following condition holds*

$$\|A(t) - A_0\| \le \alpha < \frac{1}{2\|H_0\|}, \quad \forall t > T_0,$$

where H_0 is the solution of the matrix Lyapunov equation

$$H_0 A_0 + A_0^* H_0 = -I.$$

Then the following estimate holds

$$\|y(t)\|^2 \le \mu(H_0) e^{\left(2\alpha - \frac{1}{\|H_0\|}\right)(t-s)} \|y(s)\|^2, \quad t > s,$$

where $\mu(H_0)$ is the condition number of the matrix H_0.

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On Controllability of Semilinear Stochastic Delay Control Systems

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In this work, we study the approximately controllability of a Wiener process driven semilinear stochastic delay control system with single-point delay in control in Hilbert spaces. The ideas of C_0-semigroup, controllability operator and control function are employed to establish the sufficient conditions for approximate controllability. The main result is deduced under the assumption that the associated linear deterministic control system is approximately controllable.

Keywords: approximate controllability, semilinear systems, stochastic delay control system, reachable set, control delay

1 The main results

Let X, U and E be separable Hilbert spaces, $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with normal filtration $\{\mathcal{F}_t : t \in [0,T]\}$ generated by a Q-Wiener process $\{\omega(s) : 0 \le s \le t\}$, where $Q \in \mathcal{L}(E)$ is the covariance operator with $tr(Q) < \infty$, and $L_0^2 = L^2(Q^{1/2}E; X)$. Let $L^2(\Omega, \mathcal{F}_t, X)$ be the space of square-integrable $\mathcal{F}$-measurable random variables taking values in X and $L^2_{\mathcal{F}}([-h,T], X)$ be the space of $\mathcal{F}_t$-adapted X-valued measurable and square-integrable processes. Consider the system

$$\begin{cases} dx(t) = [Ax(t) + Bu(t) + B_1 u(t-h) + f(t, x(t))]dt + \sigma(t, x(t))d\omega(t), \ \ t \in (0,T], \\ x(0) = x_0, \\ u(t) = 0, \ t \in [-h, 0], \ h > 0, \end{cases} \tag{1}$$

where $h > 0$ is constant; $A : D(A) \subset X \to X$ is generator of C_0-semigroup $\{S(t)\}_{t \ge 0}$; $B : L^2([0,T], U) \to L^2([0,T], X)$ and $B_1 : L^2([-h,T], U) \to L^2([0,T], X)$ are bounded linear operators; state function $x \in C([0,T], L^2(\Omega, \mathcal{F}_t, X))$; control $u \in L^2_{\mathcal{F}}([-h,T], U)$; drift term $f : [0,T] \times X \to X$ and the diffusion term $\sigma : [0,T] \times X \to L_0^2$ are nonlinear maps.

Assumption 1 *Maps f and σ are continuous maps satisfying Lipschitz condition in second variable and a growth condition, that is, for all $t \in [0,T]$ and $x(t), y(t) \in X$, there exists constants $L_1, L_2, L_3, L_4 > 0$, such that:*

1. $\|f(t, x(t)) - f(t, y(t))\|^2 \le L_1 \|x(t) - y(t)\|^2$.
2. $\|\sigma(t, x(t)) - \sigma(t, y(t))\|^2_{L_0^2} \le L_2 \|x(t) - y(t)\|^2$.
3. $\|f(t, x(t))\|^2 \le L_3(1 + \|x(t)\|^2)$.
4. $\|\sigma(t, x(t))\|^2_{L_0^2} \le L_4(1 + \|x(t)\|^2)$.

Under the conditions 1-4, the unique mild solution $x(t; x_0, u) \in L^2(\Omega, \mathcal{F}_t, X)$ of (1) exists, represented by the following integral equation

$$x(t; x_0, u) = \begin{cases} S(t)x_0 + \displaystyle\int_0^t S(t-s)[Bu(s) + f(s, x(s))]ds \\ \qquad\qquad + \displaystyle\int_0^t S(t-s)\sigma(s, x(s))d\omega(s), \ t \in [0,h], \\ S(t)x_0 + \displaystyle\int_0^{t-h} [S(t-s)B + S(t-h-s)B_1]u(s)ds + \displaystyle\int_{t-h}^t S(t-s)Bu(s)ds \\ \quad + \displaystyle\int_0^t S(t-s)f(s, x(s))ds + \displaystyle\int_0^t S(t-s)\sigma(s, x(s))d\omega(s), \ t \in [h,T]. \end{cases} \tag{2}$$

Definition 1. *Define the reachable set*

$$\mathcal{R}_T(f) = \left\{ x(T; x_0, u) \in L^2(\Omega, \mathcal{F}_T, X) : u \in L^2_{\mathcal{F}}([-h,T], U) \right\},$$

where

$$\begin{aligned} x(T; x_0, u) =& S(T)x_0 + \int_0^{T-h} [S(T-s)B + S(T-h-s)B_1]u(s)ds + \int_{T-h}^T S(T-s)Bu(s)ds \\ &+ \int_0^T S(T-s)f(s, x(s))ds + \int_0^T S(T-s)\sigma(s, x(s))d\omega(s). \end{aligned}$$

The deterministic semilinear delay control system associated to (1) is

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B_1 u(t-h) + f(t, x(t)), \quad t \in (0, T], \\ x(0) = x_0, \ u(t) = 0, \ t \in [-h, 0], \quad h > 0, \end{cases} \tag{3}$$

Assumption 2 *System* (3) *is approximately controllable on* $[0, T]$.

Definition 2. *System* (1) *is called approximately controllable on* $[0, T]$ *if* $\overline{\mathcal{R}_T(f)} = L^2(\Omega, \mathcal{F}_T, X)$.

Theorem 1. *Suppose assumptions 1 and 2 hold, the* C_0*-semigroup* $\{S(t) : t \in [0, T]\}$ *is compact and* f, σ *are uniformly bounded. Then, the semilinear stochastic delay control system* (1) *is approximately controllable on* $[0, T]$.

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Efficient and Robust Estimation and Test for Varying Coefficient Partially Nonlinear Model by Quantile Regression Estimation Method

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Keywords: varying coefficient partially nonlinear model, quantile regression estimation, asymptotic properties, local linear fitting, bootstrap

1 Quantile regression estimation

The varying coefficient partially nonlinear model takes the form of

$$Y = X^T a(U) + g(Z;) + \varepsilon \tag{1}$$

The quantile regression loss function is

$$Q(a(\cdot),) = \sum_{i=1}^{n} K \left\{ \rho_\tau (Y_i - X_i^T a(U_i) - g(Z_i,)) \right\}, \tag{2}$$

where $\rho_\tau(z) = z\{\tau - I(z \le 0)\}$ and $K_i = K(\frac{U_i - u_0}{h})$ is sign function.

First of all, we fix β and re-write the model (1) as follows.

$$Y_i - g(Z_i,) = X_i{}^T a(U_i) + \varepsilon_i, \quad i = 1, 2, \cdots, n. \tag{3}$$

We use the local linear fitting method to estimate the coefficient function vector $\boldsymbol{a}(\cdot)$ in (3).

$$a_j(u) \approx a_j(u_0) + a_j'(u_0)(u - u_0), \quad j = 1, 2, \cdots, q$$

Then, $\{(a_j(u_0), a_j'(u_0)), \quad j = 1, 2, \cdots, q\}$ can be estimated to minimize the following objective function.

$$\sum_{i=1}^{n} \rho_\tau(Y_i - g(Z_i,) - \sum_{j=1}^{q}[a_j(u_0) + a_j'(u_0)(U_i - u_0)X_{ij}]K_h(U_i - u_0) \tag{4}$$

Next, we can estimate β to minimize the following objective function

$$L() \equiv L(\hat{a}(\cdot),) = \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^T \hat{a}(U_i) - g(Z_i,)). \tag{5}$$

The following theorems shows the asymptotic normality of the proposed estimators.

Theorem 1. *Suppose that the matrix Σ is positive definite and the conditions $C1$-$C8$ hold. Then, $\hat{\beta}$ converges in probability to β_0. Also,*

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} N\left(0,\ \frac{\tau(1-\tau)\Sigma^{-1}}{f(0)}\right).$$

where $\xrightarrow{d}$ stands for the convergence in distribution.

Theorem 2. *Suppose that the conditions $C1$-$C8$ hold. Then, for any $u_0 \in \Omega$, we have*

$$\sqrt{nh}\left(\hat{a}^T(u_0; \hat{\beta}) - a_0^T(u_0) - 2^{-1}h^2\mu_2 a''^T_0(u_0), h[\hat{a}'^T(u_0; \hat{\beta}) - a'^T_0(u_0)]\right) \xrightarrow{d} N(\mathbf{0}, \Sigma(u_0))$$

where $(u_0) = \tau(1-\tau)f(0)^{-1}f_U(u_0)^{-1}\Gamma(u_0)^{-1}$. In particular, we have

$$\sqrt{nh}\left[\hat{a}(u_0; \hat{\beta}) - a_0(u_0) - 2^{-1}h^2\mu_2 a''_0(u_0)\right] \xrightarrow{d} N(\mathbf{0},\ \tau(1-\tau)\nu_0 f(0)^{-1}f_U(u_0)^{-1}\Gamma(u_0)^{-1}).$$

2 Testing a linear relationship of the nonparametric component

This can be achieved by testing the following hypotheses:

$$\begin{cases} H_0 : a_j(u) = a_j \ \text{ for all } \ j = 1, 2, \dots, q \\ H_1 : a_j(u) \ \text{ is varying with } u \text{ for at least one of } \ j = 1, 2, \dots, q, \end{cases} \tag{6}$$

where $a_1, a_2, \dots, a_q$ are unknown constants. Then the resulting residual sum of the quantile loss functions under the null and alternative hypotheses are, respectively,

$$RSQ(H_0) = \sum_{i=1}^{n} \rho_\tau\{Y_i - X_i^T\hat{a} - g(Z_i, \hat{})\}$$

and

$$RSQ(H_1) = \sum_{i=1}^{n} \rho_\tau\{Y_i - X_i^T\hat{a}(U_i; \hat{\beta}) - g(Z_i, \hat{})\}.$$

Based on RSQ(H_0) and RSQ(H_1), the statistic is constructed as

$$T = \frac{n}{2}\frac{\{RSQ(H_0) - RSQ(H_1)\}}{RSQ(H_1)} \tag{7}$$

The residual-based bootstrap procedure for deriving the p-value of the test is proposed as follows.

Step 1. Based on the data set $\{Y_i, U_i, X_i, Z_i\}_{i=1}^n$ and a predetermined value of the bandwidth h, compute under H_1 the residual vector $\hat{} = (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_n)^T$ and centralize the residual vector $\hat{}$ to obtain $\hat{}_c = (\hat{\varepsilon}_1 - \bar{\hat{\varepsilon}}, \hat{\varepsilon}_2 - \bar{\hat{\varepsilon}}, \dots, \hat{\varepsilon}_n - \bar{\hat{\varepsilon}})^T$ in which $\bar{\hat{\varepsilon}} = Q^{(\tau)}(\hat{})$ is τth sample quantile of $\hat{\varepsilon}_i, \quad i = 1, 2, \dots, n$. Furthermore, compute under H_0 the estimator $\hat{a}$ of a. With the estimation results under both H_0 and H_1, compute the observed value t of the statistic T by (7).

Step 2. Draw a bootstrap residual vector $\hat{}^* = (\hat{\varepsilon}_1^*, \hat{\varepsilon}_2^*, \dots, \hat{\varepsilon}_n^*)^T$ with replacement from the empirical distribution function of $\hat{}_c$.

Step 3. Generate $Y_i^* = X_i{}^T\hat{a} + g(Z_i, \hat{}) + \hat{\varepsilon}_i^*, \quad i = 1, 2, \cdots, n$, and calculate the bootstrap version T^* of the statistic T by

$$T^* = \frac{n}{2}\frac{\{RSQ^*(H_0) - RSQ^*(H_1)\}}{RSQ^*(H_1)} \tag{8}$$

Step 4. Repeat Steps 2 and 3 m times and obtain a bootstrap sample of the statistic T as $T_1^*, T_2^*, \dots, T_m^*$. The p-value is then estimated by

$$\hat{p} = \frac{\#\{T_i^* : T_i^* \geq t\}}{m} \tag{9}$$

where #A denotes the number of the elements in the set A.

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Spectral-Galerkin Method for Electromagnetic Transmission Eigenvalue Problems in Two-dimensional Anisotropic Media

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We propose an efficient spectral Galerkin method for the electromagnetic transmission eigenvalue problem in two-dimensional anisotropic media. We formulate the associated variational form, discuss the spectral Galerkin approximation, and prove the error estimates for eigenvalues and eigenfunctions by using the spectral theory of compact operators and the approximation properties of orthogonal projection operators in nonuniform weighted Sobolev spaces. Numerical examples are given to demonstrate the effectiveness of the proposed spectral method.

Keywords: electromagnetic transmission eigenvalue problem, spectral method, anisotropic media, error estimates.

The transmission eigenvalue problem holds theoretical and practical significance in inverse scattering theory since transmission eigenvalues have information about the material properties of the scattering object. Since the spectral method can obtain high-accuracy numerical solutions with low computational cost, there is a lot of research on numerical solutions of the transmission eigenvalue problem based on this method. However, up to date, numerical methods based on spectral methods have been studied only for isotropic media [1,2,3]. We will study spectral methods for the two-dimensional electromagnetic transmission eigenvalue problem for anisotropic media.

In this paper, we assume the scatterer is an infinitely long dielectric cylinder with axis in the z-direction and let the incident electromagnetic field be a plane wave propagating in a direction perpendicular to the cylinder. Let $D \subset R^2$ with piecewise smooth boundary ∂D be a cross section of the cylinder and let the exterior domain be a connected. The dielectric cylinder is assumed to consist of an orthogonal media. Then, the electromagnetic transmission eigenvalue problem for an inhomogeneous anisotropic media is formulated as follows: Find $k^2 \in C$ and a nonzero $(u, v) \in H^1(D) \times H^1(D)$ such that

$$\begin{cases} \cdot Au + k^2 nu = 0, \quad x \in D, \\ \triangle v + k^2 v = 0, \quad x \in D, \\ u - v = 0, \quad x \in \partial D, \\ \dfrac{\partial u}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu}, \quad x \in \partial D, \end{cases} \tag{1}$$

The variational form for the transmission eigenvalue problem (1) can be formulated as follows: Find $(\lambda, U) \in C \times V, U \neq 0$ satisfying

$$a(U, \Psi) = \lambda b(U, \Psi), \quad \forall \Psi \in V \tag{2}$$

where

$$a(U, \Psi) = (Au, w) + (nu, w) - (v, z) - (v, z),$$
$$b(U, \Psi) = (nu, w) - (v, z),$$

Let us introduce the finite dimensional subspaces V_N and W_N of V and W as follows:

$$V_N := V \cap ((P_N(D))^2 \times (P_N(D))^2), W_N := W \cap ((P_N(D))^2 \times (P_N(D))^2).$$

Then the spectrum-Galerkin approximation to formula (2) is a problem of finding $\lambda_N = k_N^2 + 1 \in C$ satisfying

$$a(U_N, \Psi_N) = \lambda_N b(U_N, \Psi_N), \quad \forall \Psi_N \in V_N. \tag{3}$$

Theorem 1 There exists a positive constant C such that

$$\epsilon_N \leq cN^{1-m}(\|\partial_x^m u\|_{\omega^{m,m}} + \|\partial_x^m v\|_{\omega^{m,m}}) \tag{4}$$

$$\epsilon_N^* \leq cN^{1-m}(\|\partial_x^m u^*\|_{\omega^{m,m}} + \|\partial_x^m v^*\|_{\omega^{m,m}}). \tag{5}$$

Then we have the following error estimates of the approximation of the eigenvalues and eigenfunctions.

Theorem 2 There exists a positive constant C such that

$$\begin{cases} \delta(R(E), R(E_N)) \leq CN^{1-m}\epsilon_N \\ |\tau - (\dfrac{1}{g}\sum_{j=1}^{g} \tau_{j,N}^{-1})^{-1}| \leq CN^{2(1-m)}\epsilon_N \epsilon_N^* \\ |\tau - \tau_{j,N}^{-1}| \leq CN^{\overline{\dfrac{2(1-m)}{\alpha}}}[\epsilon_N \epsilon_N^*]. \end{cases} \tag{6}$$

Theorem 3 Let λ_N be an eigenvalue of (3.9) and $\lim_{N\longrightarrow\infty} \lambda_N = \lambda$. Let us assume that for every N there is a some positive constant $k \leq \alpha$ such that $\|(u_N, w_N)\|_{X^d} = 1$ and $(\lambda_N^{-1} - K)^k (u_N, w_N) = 0$ are satisfied.hen there exists a vector (u, w) satisfying $(\lambda^{-1} - K)^l (u, w) = 0$ for an integer l with $k \leq l \leq \alpha$, and the following inequality holds:

$$\|(u, w) - (u_N, w_N)\|_V \leq C(N^{1-m}\epsilon_N)^{\overline{\dfrac{l-k+1}{\alpha}}}. \tag{7}$$

Numerical experimental results have verified the effectiveness of the proposed spectral method.

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A Method for Searching Polynomial Coefficients Using a Pseudorandom Number Generator in the Inverse Problem for the Grad–Shafranov Equation

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A numerical approach to solving the inverse problem for the Grad–Shafranov equation is presented. The approach is based on recovering the coefficients of polynomial approximations. A method for searching polynomial coefficients is proposed that combines a pseudorandom number generator with the conjugate gradient method. At each step of the algorithm, pairs of third-degree polynomials are generated and then numerically tested according to distinguishability and consistency criteria with the prescribed data. A software implementation was developed, and a series of computational experiments was carried out. As a result, new pairs of polynomials satisfying the conditions of the problem were found.

Keywords: polynomial approximation, pseudorandom number generator, conjugate gradient method, inverse problem for the Grad–Shafranov equation

In this paper, we study the inverse problem for the simplified Grad–Shafranov equation:

$$\begin{cases} \Delta u = f(u(x,y)), & (x,y) \in \omega, \\ u = 0, & (x,y) \in \gamma = \partial\omega. \end{cases} \tag{1}$$

The problem of reconstructing the distribution $f(u(x,y))$ was discussed in [1]. However, in [2] the question was raised as to whether the obtained solutions may fail to coincide with the actual distribution. It was proposed to find at least two essentially different distributions $f_u^1 = f^1(u_1)$ and $f_u^2 = f^2(u_2)$ satisfying problem (1) and the conditions

$$\begin{cases} \max\limits_{P\in\gamma} \left| \dfrac{\partial u_2}{\partial n}(P) - \dfrac{\partial u_1}{\partial n}(P) \right| \le \lambda, \\ \left| \dfrac{f_u^1 - f_u^2}{\max\{f_u^1, f_u^2\}} \right| \ge 0.1, \\ f_u^{j\,\mathrm{ef}} = \max\limits_{(x,y)\in\omega} \left| f_u^j(x,y) \right|, \qquad j = 1,2. \end{cases} \tag{2}$$

where $\lambda > 0$ is a small parameter and n is the outward normal to γ. In [3], the possibility of existence of essentially different distributions f_u^1 and f_u^2 for problem (1) was shown in the class of polynomial functions. In the present work, third-degree polynomials are considered:

$$f^k(u) = a_1^k + a_2^k u + a_3^k u^2 + a_4^k u^3, \quad k = 1,2. \tag{3}$$

The method used to search for the polynomial coefficients is based on a pseudorandom number generator and the conjugate gradient method. At each iteration of the outer loop, two arrays of coefficients defining third-degree polynomials are generated. For each pair of coefficient arrays, the functions f^1 and f^2 are reconstructed, after which their essential difference is checked. If this condition is satisfied, the normal derivatives of the solutions u_1 and u_2 are computed, and then the essential-difference condition based on the maximum discrepancy of the normal derivatives is verified. Pairs of polynomials satisfying both conditions are stored in the result array.

For the software implementation, the rectangular domain $\omega = \{(x, y) \colon |x| < 3/2,\ |y| < 1\}$ was chosen. Taking central symmetry into account, it is sufficient to consider the first quarter of the domain, $\omega_1 = \{(x, y) \colon 0 < x < 3/2,\ 0 < y < 1\}$. With the search interval for the coefficients restricted to $a_i \in [-1.00; 16]$, 10 new pairs of polynomials satisfying condition (2) were found.

The computational experiments showed that the proposed algorithm makes it possible to find new pairs of polynomials satisfying the prescribed distinguishability criteria. The obtained results confirm the applicability of the combination of a pseudorandom number generator and the conjugate gradient method to the study of the inverse problem for the Grad–Shafranov equation.

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Beta Marshall-Olkin Extended Generalized Rayleigh Distribution and its Maximum Likelihood Estimation

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We introduce special model-Beta Marshall-Olkin generalized Rayleigh distribution. Beta Marshall-Olkin generalized Rayleigh distribution is a new distribution model with 5 parameters. We derive new probability density function and hazard rate function of Beta Marshall-Olkin extended generalized Rayleigh distribution with 5 parameters. And we suggest the maximum likelihood estimators of medel

parameters by numerical method. In addition, we illustrate the usefulness of the new distribution model through the alumina data comparing with competing models such as Generalized Rayleigh (GR) distribution, Beta GR distribution, Marshall-Olkin GR distribution and Marshall-Olkin Extended Gumbel Type-II distribution. We have demonstrated as better fit than Marshall-Olkin Extended Gumbel Type-II distribution which has been the best model for alumina data.

Keywords: Marshall-Olkin generator, beta Marshall-Olkin family, maximum likelihood estimation

1 Introduction

By using Marshall-Olkin generator technique, we obtain a three-parameter Marshall-Olkin extended generalized Rayleigh distribution with one extra parameter. The cdf of the three-parameter Marshall-Olkin generalized Rayleigh distribution is expressed as

$$G(x;\alpha) = \frac{F(x)}{1-(1-\alpha)\bar{F}(x)}, \quad x \in R, \quad \alpha > 0 \tag{1}$$

If α=1 then above cdf (1) reduces to generalized Rayleigh distribution. And incomplete gamma function $\gamma(\lambda+1,\ \theta x^2)$ is expanded as the form of following.

$$\gamma(\lambda+1,\theta x^2) = (\theta x^2)^{\lambda+1} \sum_{i=0}^{\infty} \frac{(-1)^i(\theta x^2)^i}{i!(\lambda+1+i)}$$

We can also derive the pdf of three-parameter Marshall-Olkin extended generalized Rayleigh distribution from cdf (1)

Theorem 1. The pdf of the beta Marshall-Olkin extended generalized Rayleigh distribution is a function with five parameters, which is

$$f(x;a,b,\alpha,\theta,\lambda) = \frac{2\alpha^b\theta^{\lambda+1}x^{2\lambda+1}e^{-\theta x^2}\big(G_{\theta,\lambda}(x)\big)^{a-1}\big(1-G_{\theta,\lambda}(x)\big)^{b-1}}{B(a,b)\Gamma(\lambda+1)\big(\alpha+\bar{\alpha}G_{\theta,\lambda}(x)\big)^{a+b}} \tag{2}$$

where $x \in R, a > 0, b > 0, \alpha > 0, \theta > 0, \lambda > -1$.

Corollary 2. The hazard rate function of the three-parameter Marshall-Olkin generalized Rayleigh distribution is given by

$$h(x;\alpha,\theta,\lambda) = \frac{2\theta^{\lambda+1}\cdot x^{2\lambda+1}e^{-\theta x^2}}{(\alpha\Gamma(\lambda+1)+\bar{\alpha}\gamma(\lambda+1,\theta x^2))\left(1-\frac{\gamma(\lambda+1,\theta x^2)}{\Gamma(\lambda+1)}\right)} \tag{3}$$

2 Maximum Likelihood Estimation

Here we determine the MLE's of the model parameters of the new distribution model. Let x_1, x_2, ..., x_n be observed samples from the BMOEGR distribution(2). And let φ=(a, b, α, θ, λ)^T be the 5×1 parameter vector. The total log-likelihood function for φ=(a, b, α, θ, λ)^T is given by

$$l(\varphi) = nlog2\ - nlog\{\ B(a,b)\} - nlog\{\ (\lambda+1)\} + nblog\alpha\ + n(\lambda+1)log\theta +$$

$$+(2\lambda+1)\sum_{i=1}^{n} logx_i\ - \theta\sum_{i=1}^{n} x_i^2 + (a-1)\sum_{i=1}^{n} log\{\ G_{\theta,\lambda}(x_i)\} + (b-1)\sum_{i=1}^{n} log(\ 1-G_{\theta,\lambda}(x_i))$$

$$- (a+b)\sum_{i=1}^{n} log\{\ \alpha + \bar{\alpha}G_{\theta,\lambda}(x_i)\} \tag{4}$$

The estimators of the parameters $(a,\ b,\ \alpha,\ \theta,\ \lambda)^T$ are determined by solving the maximization problem of log-likelihood function (3).

Generally, there is no closed-form expression for the maximum likelihood estimators. In addition, since the number of parameters is five, we can find the maximum likelihood estimator of the remaining parameters given a few parameters by numerical calculation. G.M. Cordeiro et al. [3] obtained the maximum likelihood estimates for the parameters of the beta-generalized Rayleigh distribution using the Newton Raphson iteration algorithm. Also, S.M.T.K. MirMostafaee et al. [4] found the maximum likelihood estimator for the parameters of the Marshall-Olkin generalized Rayleigh distribution. The parameter estimates of the proposed distribution were compared to the Marshall-Olkin generalized Rayleigh distribution and the parameter estimates of the beta generalized Rayleigh distribution to compute the fitness of the model.

Table 1. MLEs of the model parameters for the stress level data

Model	a	b	α	θ	λ	AIC	BIC
GR	1	1	1	0.22162	3.3742	356.97	370.86
MOEGR	1	1	10.363	0.23648	1.3687	345.25	359.15
BGR	0.54209	38.052	1	0.048689	3.4812	346.96	360.86
BMOEGR	0.32525	0.23836	39.764	1.0137	9.879	341.86	355.75
MOEGT-II	-	-	-	-	-	345.03	356.15

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Study on the Optimization of Levitation Force with Changing Superconductor Geometry in Three-Surface Levitated Superconducting Magnetic Bearing

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In this paper, the levitation force variation with the superconductor geometry by experimental design $L_3 2(2^1 \times 4^9) \otimes L_8(2^7)$ in a three surface levitation superconducting magnetic bearing is considered and its shape with a larger levitation force is estimated. In addition, the significance of the factors on the change of levitation force is analyzed using the Taguchi method, and the corresponding optimum parameters are obtained.

Keywords: three surface levitation, superconducting magnet bearing (SMB), H-formulation, Taguchi method

1 Model of superconducting magnetic bearing and Simulation

In this paper, we consider the levitation force variation with the superconductor geometry in a three-surface levitation superconducting magnetic bearing.(Fig. 1)

The geometrical parameters for the three surface levitation superconducting magnetic bearing are shown in Table 1.

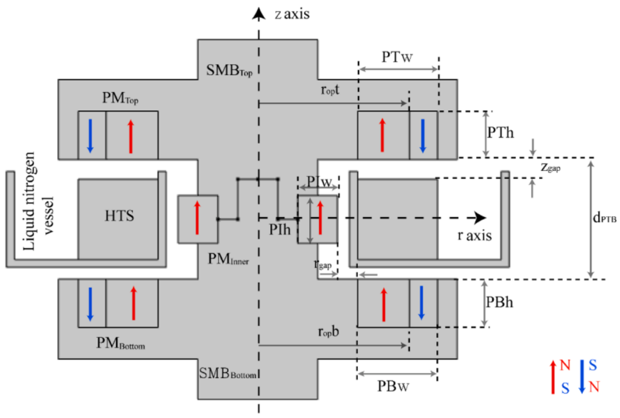

Figure 1: The schematic of TSL-SMB

Table 1: Geometrical parameters for three surface levitation superconducting magnetic bearing

Symbol	Definition	Outer diameter×Inner diameter×Height, mm
HTS	Ringtype HTS bulk	(70+W_HTS)×(70W_HTS)×H_HTS
PMI	Inner PM on the ringtype HTS bulk	40×20×10
PMTI	TopInner PM on the ringtype HTS bulk	r_{ORTI}×50×10
PMTO	Top-Outer PM on the ringtype HTS bulk	90×r_{ORTI}×10
PMO	Outer PM on the ringtype HTS bulk	120×100×10
PMBI	BottomInner PM on the ring-type HTS bulk	r_{ORBI}×50×10
PMBO	BottomOuter PM on the ringtype HTS bulk	90×r_{ORBI}×10

The levitation force is expressed by Lorentz's force formulation as:

$$F = 2\pi\mu_0 \int_S rJ_\phi H_r dS \tag{1}$$

where J_φ represents the current density in the azimuthal direction, H_r is the radial magnetic field strength and S means the rz plane cross-section area in HTS, and r is its radius of the ring-type HTS bulk.

The numerical simulation is performed by COMSOL Multiphysics. For numerical simulation we used experimental planning $L_{32}(2^1 \times 4^9) \otimes L_8(2^7)$ by Taguchi method

Table 2 shows the factors and levels considered in three surface levitation superconducting magnetic bearing

Table 2: Factors and levels considered in three surface levitation superconducting magnetic bearing

No	Factor	Definition	Level, *mm* 1	2	3	4
1	gap	Difference between the outer and the inner diameter of the HTS	3.5	5	-	-
2	delw	Inner diameter of the outer permanent magnet of the PM located at the upper limit	0	2	4	8
3	Piw	Inner diameter of the outer permanent magnet located at the lower limit	10	12	14	16
4	Pih	Height of the Inner Permanent Magnet	16	18	20	22
5	PTw	Difference between the inner and outer inclinations of the upper permanent magnet	20	23.5	27	30
6	PTh	Height of the upper permanent magnet	15	17	19	21
7	PBw	Difference between the inner and outer inclinations of the lower permanent magnet	20	23.5	27	30
8	PBh	Height of the lower-located permanent magnet	15	17	19	21
9	ratT	Ratio of the upper-positioned inner and outer permanent magnets	0.2	0.4	0.6	0.8
10	ratB	Ratio of the inner and outer permanent magnets located downward	0.2	0.4	0.6	0.8
11	Ptin	Magnetization direction of the upper inner permanent magnet	1	2	-	-
12	Ptout	Magnetization direction of the upper outer permanent magnet	1	2	-	-
13	Pbin	Magnetization direction of the lower inner permanent magnet	1	2	-	-
14	Pbout	Magnetization direction of the lower outer permanent magnet	1	2	-	-

The parameters of the optimal level scheme where the levitation force is maximized are shown in Table 3.

2 Result analysis and Conclusion

In a three surface levitation superconducting magnetic bearing, the levitation force varies with the superconductor geometry, so the optimal geometrical parameters must be set. Of course, in this paper, the complete optimization of the quadrature levitated superconducting magnetic bearing was not carried out, but the geometry for enhancing the levitation force in the superconducting magnetic bearing was predicted, because some of the entanglements between the factors located in the inner orthogonal array and the entanglement of the three factors located in the outer orthogonal array were not considered and the response function was not considered.

In the future, more superconducting magnetic bearings will be proposed, which will contribute to cleaner renewable energy use.

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Table 3: Optimum level scheme and Parameters

Factor	Optimum level scheme	
	Level	Size, mm
gap	1	3.5
delw	4	8
Piw	1	10
Pih	2	18
PTw	4	30
PTh	3	19
PBw	3	27
PBh	4	21
ratT	2	0.4
ratB	1	0.2
Ptin	2	2
Ptout	1	1
Pbin	1	1
Pbout	1	1
levitation force, N		406.05

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Investigation of Differential Equations with Increasing Coefficients by Means of Transformation Operators

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The Stark effect is the shifting and splitting of spectral lines of atoms and molecules due to presence of an external electric field. The effect is named after Stark, who discovered it in 1913. The Stark effect has been of marginal benefit in the analysis of atomic spectra, but has been a major tool for molecular rotational spectra. The perturbation theory of the Stark effect is of particular interest. The application of transformation operators to the perturbation theory of linear operators is well known (see [1], [2] and the references therein). Not that in [6]–[8] some related problems were considered.

We consider the differential equation

$$-y'' + xy + p(x)\,y + q(x)\,y = \lambda y, \quad -\infty < x < +\infty, \ \lambda \in C. \tag{1}$$

Where real potentials $p(x)$ and $q(x)$ satisfy the conditions

$$p(x) = \begin{cases} \alpha_+, & x \geq 0, \\ \alpha_-, & x < 0, \end{cases} \tag{2}$$

$$q(x) \in C(-\infty, +\infty) \quad \int_{-\infty}^{+\infty} |xq(x)|\, dx < \infty. \tag{3}$$

In the present paper, using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. The results obtained can be used to solve inverse spectral problems for an equation (1). Some questions of the spectral theory of the one-dimensional Stark equation were studied in [3]–[4].

In what follows, we deal with special functions satisfying the Airy equation

$$-y'' + zy = 0.$$

It is well known (e.g., see [5]) that this equation has two linearly independent solutions $Ai(z)$ and $Bi(z)$ with the initial conditions

$$Ai(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)},\ Ai^{'(0)} = \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)},$$

$$Bi(0) = \frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)},\ Bi^{'(0)} = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}.$$

The Wronskian $\{Ai(z), Bi(z)\}$ of these functions satisfies

$$\{Ai(z), Bi(z)\} = Ai(z), Bi'(z) - Ai'(z)\, Bi(z) = p^{-1}.$$

Both functions are entire functions of order $\frac{3}{2}$ and type $\frac{2}{3}$. Note that the functions $Ai(x-\lambda)$, $Ai(x-\lambda) - iBi(x-\lambda)$ satisfy the relations (see [2]) $Ai(x-\lambda) \in L_2(0, +\infty)$, $Ai(x-\lambda) - iBi(x-\lambda) \in L_2(-\infty, 0)$ for $Im\lambda \geq 0$.

In what follows we will need special solutions of the unperturbed equation

$$-y'' + xy + p(x)\, y = \lambda y,\ \ -\infty < x < +\infty,\ \lambda \in C. \tag{4}$$

Theorem 1. *If the potentials $p(x)$ and $q(x)$satisfy the conditions (2),* (3) *then for any λ from the closed upper half-plane equation* (1) *has a solution $f_+(x,\lambda)$ that can be represented in the form*

$$f_+(x,\lambda) = \lambda_+(x,\lambda) + \int_z^{\infty} K_+(x,t)\, \lambda_+(t,\lambda) dt,$$

where kernel $K_+(x,t)$ is continuous function and satisfies relations

$$K_+(x,t) = O\left(s_+\left(\frac{x+t}{2}\right)\right), \qquad x+t \to \infty,$$

$$K_+(x,x) = \frac{1}{2}\int_x^{\infty} [p(t) - a_+ + q(t)]\, dt.$$

Theorem 2. *If the potentials $p(x)$ and $q(x)$ satisfy the conditions (2), (2), then, for any λ from the closed upper half-plane, equation* (1) *has a solution $f_-(x,?)$representable as*

$$f_-(x,\lambda) = \lambda_-(x,\lambda) + \int_{-\infty}^{x} K_-(x,t)\, \lambda_-(t,\lambda) dt.$$

where the kernel $K_-(x,t)$is continuous function and satisfy the following conditions

$$K_-(x,t) = O\left(s_-\left(\frac{x+t}{2}\right)\right), \qquad x+t \to -\infty,$$

$$K_-(x,x) = \frac{1}{2}\int_{-\infty}^{x} [p(t) - a_- + q(t)]\, dt.$$

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Time-Optimal Problem for Schroedinger Equation*

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We consider a time-optimal problem on the Lie group $SU(2)$ for a Schroedinger equation that arises as a model of a 2-level closed quantum system (one isolated qubit) under the action of a coherent control. We solve the question of existence of optimal trajectories and obtain their description.

Keywords: time-optimal problem, Schroedinger equation, geometric control theory

Consider right-invariant vector fields on the Lie group $SU(2)$ of the form $X_j = i\sigma_j q$, $j = 1, 2, 3$, where σ_j are the Pauli matrices. We study the time-optimal problem

$$\begin{aligned}
&\dot{q} = -X_3 - u(v_1 X_1 + v_2 X_2), \qquad q \in G = SU(2), \quad u \in U,\\
&q(0) = q_0 = Id, \qquad q(t_1) = q_1,\\
&t_1 \to \min,
\end{aligned}$$

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where $U = \mathbb{R}$ or $U = [-C, C]$ for a given $C > 0$, and $(v_1, v_2) \in \mathbb{R}^2 \ \{(0,0)\}$.

The main result is the following

Theorem 1. *Let $q_1 = e^{-\varphi X_3} \in SU(2)$, $\varphi \in (0, \frac{\pi}{2}]$. Then the control $u(t) \equiv 0$, $t \in [0, \varphi]$, is optimal with $U = \mathbb{R}$ or $U = [-C, C]$.*

On Radially Symmetric Solutions for an Elliptic Equation with p(|x|)-Laplacian

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Keywords: $p(|x|)$-Laplace equation, radially symmetric solution

Let us consider the following Dirichlet problem

$$-\,div(|\nabla u|^{p(|x|)-2}\nabla u) = F(x, u, \nabla u) \quad \text{in} \quad B_R \subset \mathbb{R}^n, \quad u = 0 \quad \text{on} \quad \partial B_R, \tag{1}$$

where B_R is a ball of radius R, ∂B_R is the boundary of B_R, $p(|x|) > 2$. The presence of gradient terms in the equation significantly complicates the application of variational methods to study the solvability of the problem (1). We also note that the equation in (1) has an important property: it is not scale-invariant, which makes inapplicable many traditional methods for studying solvability and analyzing the qualitative properties of solutions.

We are interested in the existence of bounded radially symmetric solutions of problem (1). We will assume that the function $F(x, u, \nabla u)$ can be represented as $F(r, u, u_r)$ under the change of variables $r = |x|$. In what follows, we will denote the derivative of the function u with respect to the variable r as u'. It is well known that the bounded radially symmetric solution of (1) satisfies the equation

$$-\,(|u'|^{p(r)-2}u')' - \frac{n-1}{r}|u'|^{p(r)-2}u' = F(r, u, u'), \quad r \in (0, R), \tag{2}$$

and boundary conditions

$$u'(0) = 0, \quad u(R) = 0. \tag{3}$$

One of the standard conditions in the study of the solvability of the problem (1) is the Bernstein–Nagumo condition imposed on the function $F(r, u, u')$,

$$|F(r, u, q)| \le c\left(1 + |q|^{p(r)}\right) \quad \text{for} \quad (r, u, q) \in [0, R] \times [-M, M] \times \mathbb{R}, \tag{4}$$

with some constant c, provided that the solution satisfies the condition $\max|u| \le M$ with some constant M. We are interested in the solvability of (1) without any smallness conditions in the case where the function $F(r, u, q)$ has arbitrary growth with respect to the variable q.

We do not impose any restrictions on the right-hand side of (2) that would guarantee the sign certainty of both the solution itself and its derivative. Thus, we seek its solutions in the class of weak Sobolev solutions.

Under certain conditions on the right-hand side, related to monotonicity in the variable u, replacing (4), it is possible to prove the existence of a weak Sobolev solution that has a classical derivative continuous in the sense of Hölder.

The result presented in the thesis will be published in [1].

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The Fixed Points of the Dynamical System: p-adic Potts Model with an External Field*

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In this paper, we study the fixed points of the dynamical system for the p-adic Potts model with an external field. We find three fixed points for the corresponding operator and describe their dynamical behavior.

Keywords: p-adic Potts model with an external field, fixed point, attracting, repelling, indefferent

1 The main results

The Potts model is a fundamental model in statistical mechanics that extends the Ising model by allowing spins to take more than two distinct states. Owing to its rich mathematical structure and wide applicability to various physical systems, the Potts model has attracted considerable attention in recent years. To study Gibbs measures for the mathematical statistical models is one of the main problems of statistical mechanics. [1]

We consider the following map $f : \mathbb{Q}_p \to \mathbb{Q}_p$, where

$$f_{\theta,\eta}(h) = \left(\frac{(\theta + q - 2)h + \eta}{(q-1)h + \theta\eta} \right)^k \tag{1}$$

To find translation invariant p-adic quasi Gibbs measures for the Potts model with an external field is equivalent to determining the fixed points of (1) (see [2]), that is,

$$h = f_{\theta,\eta}(h). \tag{2}$$

Let $x^{(0)}$ be a fixed point of an analytic function $f(x)$, and let

$$\lambda = \frac{d}{dx} f(x^{(0)}).$$

- The fixed point $x^{(0)}$ is called *attractive* if $0 \leq |\lambda|_p < 1$.
- The fixed point $x^{(0)}$ is called *indifferent* if $|\lambda|_p = 1$.
- The fixed point $x^{(0)}$ is called *repelling* if $|\lambda|_p > 1$ [3].

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The following theorem provide the number of the fixed points of $f_{\theta,\eta}(h)$.

Theorem 1. Let $p > 3$, $|q|_p = 1$, $\mathcal{N}$ be the cardinality of the set of the fixed points of $f_{\theta,\eta}(h)$. Then we have

$$\mathcal{N} = \begin{cases} 3, & \text{if } \sqrt{1-q} \equiv 1 (\operatorname{mod} p); \\ 1, & otherwise. \end{cases}$$

Theorem 1 satisfies in the case: $p > 3$, $|q|_p = 1$. In the following result gives the number of the fixed points in other conditions.

Theorem 2. For the operator $f_{\theta,\eta}(h)$, the following assertions hold:

1. If $p = 3$, $|q|_3 = 1$, then $f_{\theta,\eta}(h)$ has a unique attracting fixed point.
2. If $p \geq 3$, $|q|_p < 1$, then $f_{\theta,\eta}(h)$ does not have any fixed point.

Note that, if $p > 3$, $|q|_p = 1$, $\sqrt{1-q} \equiv 1 (\operatorname{mod} p)$, there are three fixed points for the operator $f_{\theta,\eta}(h)$. Now, we give the behaviour of the fixed points.

Theorem 3. Let $p > 3$, $|q|_p = 1$, $\sqrt{1-q} \equiv 1 (\operatorname{mod} p)$, h_0, h_1, h_2 be the fixed points of $f_{\theta,\eta}$. Then the following assertions hold:

1. The fixed point h_0 is attracting point.
2. If $|\theta - 1|_p \geq |\eta - 1|_p$ or $|\theta - 1|_p < |\eta - 1|_p < \sqrt{|\theta - 1|_p}$, then h_1, h_2 are repelling fixed points;
3. If $|\eta - 1|_p = \sqrt{|\theta - 1|_p}$, then h_1, h_2 are indifferent fixed points;
4. If $\sqrt{|\theta - 1|_p} < |\eta - 1|_p$, then h_1, h_2 are attracting fixed points;

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Relaxed Solutions for Controlled Nonconvex Sweeping BV-Processes

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This paper addresses an optimal impulsive control problem for a nonconvex sweeping process. The moving set depends on a control measure. Solutions are defined as graph completions of BV-functions. The approximation results between discontinuous solutions and classical absolutely continuous ones are established, and the existence of an optimal solution is proved.

Keywords: measure-driven sweeping process, impulsive control, graph completion, space-time reparametrization, existence of minimizer

We consider an optimal impulsive control problem for a system governed by a nonconvex sweeping process. This problem is an impulsive relaxation of the optimal control problem (P_0):

$$J_0 = l(x(b), y(b)) \to \inf,$$

$$\dot{x}(t) = f(t, x(t), y(t)) + G(t, x(t))v(t), \qquad x(a) = x_0, \tag{1}$$

$$v(t) \in K \qquad \text{for a.e. } t \in T. \tag{2}$$

Here, $T = [a, b]$ is a given time interval, $v(\cdot) \in L^\infty(T, \mathbb{R})$ is a control function such that $v(t) \in K$ for a.e. $t \in T$, where K is a closed convex cone from $\mathbb{R}^m$, $x(\cdot)$ and $y(\cdot)$ are absolutely continuous functions, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^r$, x_0 is a given initial state from $\mathbb{R}^n$. The function $y(\cdot)$ is a solution to the differential inclusion (sweeping process):

$$-\dot{y}(t) \in N^P_{C(t,x(t))}(y(t)) \ \text{ for a.e. } t \in T, \tag{3}$$

$$y(a) = y_0 \in C(a, x_0), \tag{4}$$

where $N^P_C(y)$ is the proximal normal cone to the set C at the point y. For the case when C is a convex set, the normal cone $N_C(y)$ is understood in the sense of the convex analysis.

The control system (1) and consequently the moving set C depend on a control measure, which leads to discontinuities in the state trajectories $x(\cdot)$, $y(\cdot)$. Optimal absolutely continuous solutions may fail to exist, and minimizing sequences of conventional controls converge to limits of bounded variation that involve Dirac impulses. To overcome this issue, we introduce a relaxation framework based on graph completions of BV-functions and impulsive controls. A space-time transformation converts the problem into an equivalent auxiliary control system with Lipschitz data, for which standard existence results apply. The approximation theorems establish that every relaxed solution can be approximated by classical solutions of the original problem, and that every bounded sequence of classical solutions converges to a relaxed one. Moreover, the approximation results show that the relaxed problem is the closure of the original one in the graph topology. Under a natural uniform integrability condition on the controls, we prove the existence of an optimal relaxed solution. The proposed approach provides a rigorous foundation for necessary optimality conditions and numerical methods in impulsive control of sweeping processes.

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Composition Operators on Sobolev Spaces in Metric Measure Spaces*

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We investigate a description of homeomorphisms between metric measures spaces that induce a bounden composition operator between Sobolev spaces. The description is obtained without any additional assumptions on homeomorphism or metric measure space.

Keywords: composition operator, metric measure space, mapping of Reshetnyak class, Sobolev space

We say that a triple (X, d, μ) is a *metric measure space*, if (X, d) is a separable metric space and μ is a nontrivial regular Borel measure. Also we require that μ is finite on the bounded subsets in X. We do not require a completeness of a metric space.

Let $\mathcal{F}(X)$ and $\mathcal{G}(Y)$ be (semi)normed functional spaces on X and Y respectively. We say that homeomorphism $\varphi\colon X \to Y$ between metric measure spaces (X, d, μ) and (Y, ρ, ν) *induces a bounded composition operator*

$$\varphi^*\colon \mathcal{G}(Y) \to \mathcal{F}(X)$$

by the change of variable formula if, for every $u \in \mathcal{G}(Y)$, the composition $\varphi^* u := u \circ \varphi$ is well-defined, $\varphi^* u \in \mathcal{F}(X)$, and there is a nonnegative constant K (independent of u) such that

$$\|\varphi^* u\|_{\mathcal{F}(X)} \leq K \|u\|_{\mathcal{G}(Y)}.$$

The smallest constant K above is the *norm* of the composition operator φ^*. The norm is denoted by the symbol $\|\varphi^*\|$.

Let us introduce the key concepts. *The Dirichlet space* $D^{1,q}(X)$, $1 \leq q < \infty$, consists of measurable functions u such that the inequality

$$|u(x) - u(y)| \leq \int_\gamma g \, ds$$

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holds for M_q-a.e. curves γ with endpoints x, y, where $g \in L^q(X)$ is a function (independent of γ). We say that g is an *upper gradient of* u. *The norm* of u in $D^{1,q}(X)$ is the infimum of the L^q-norms of upper gradients of u. This infimum is attained at a function $|\nabla u|_q$. Let us note that in the Euclidean space $\mathbb{R}^n$ the space $D^{1,p}$ coincides with the standard homogeneous Sobolev space $L^{1,p}$. A comprehensive introduction to the theory of Sobolev spaces on metric measure spaces can be found in [1]. We say what $u \in D^{1,q}(X) \cap \mathrm{Lip}(X)$, if u belongs to the space $D^{1,q}(X)$ and there exists a constant $L \geq 0$ such that the inequality

$$|u(x) - u(y)| \leq Ld(x, y)$$

holds for all x, $y \in X$.

We say that a measurable mapping $\varphi\colon X \to Y$ *belongs to the Reshetnyak class* $D^{1,q}(X;Y)$ if, for every $z \in Y$, the function $[\varphi]_z = \rho(z, \varphi(\cdot))$ belongs to $D^{1,q}(X)$ and there exists a majorant $w \in L^q(X)$ (independent of z) such that the inequality $|\nabla[\varphi]_z|_q \leq w$ holds μ-a.e. in X. *The upper gradient* $|D\varphi|_q$ is the L^q-minimal majorant w. This definition was introduced in [2] in the case $X = \mathbb{R}^n$. Here and henceforth, the Jacobian of a homeomorphism $\mathcal{J}\varphi$ is a density of the measure $\nu \circ \varphi$ with respect to the measure μ. The symbol Z_φ denotes the set of zeros of $\mathcal{J}\varphi$, and Σ_φ is the singularity set of φ. We say that a homeomorphism $\varphi \in D^{1,q}_{\mathrm{loc}}(X;Y)$ *has a finite distortion* if $|D\varphi|_q = 0$ μ-a.e. on Z_φ. For a mapping $\varphi \in D^{1,q}_{\mathrm{loc}}$ with a finite distortion we introduce an *operator distortion function* $K_{q,p}(x, \varphi)$ equals $|D\varphi|_q(x)/(\mathcal{J}\varphi(x))^{1/p}$, if $x \notin Z_\varphi$, and 0 otherwise.

The main result is the following description, see [3,4,5].

Theorem 1. *Let* (X, d, μ) *and* (Y, ρ, ν) *be metric measure spaces. Let also* $\varphi\colon X \to Y$ *be a homeomorphism. The homeomorphism* ϕ *induces a bounded composition operator*

$$\varphi^*\colon D^{1,p}(Y) \cap \mathrm{Lip}(Y) \to D^{1,q}(X), \quad 1 \leq q \leq p < \infty,$$

iff (*a*) $\varphi \in D^{1,q}_{\mathrm{loc}}(X;Y)$, (*b*) φ *has a finite distortion, and* (*c*) $K_{q,p}(\cdot, \varphi) \in L^\kappa(X)$, *where* $\kappa = \infty$, *if* $q = p$, *and* $1/\kappa = 1/q - 1/p$, *if* $q < p$.

Moreover, an equality $\|\varphi^*\| = \|K_{q,p}(\cdot, \varphi)\|_{L^\kappa(X)}$ *holds.*

Result of the theorem 1 consists with the previous know descriptions in model cases: Euclidean space, a Carnot group, and a Riemannian manifold.

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An Optimal Control Problem for Hyperbolic Equations

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This paper considers an optimal control problem for a hyperbolic oscillation equation where the control function appears on the right-hand side of the equation. An existence theorem for optimal control is proved, and a necessary condition for optimality is derived.

Keywords: hyperbolic equation, optimal control, optimality condition

1 The main results

Let the controlled process be described by a hyperbolic equation

$$\frac{\rho}{g}h\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial y}\right)=v\left(x,y,t\right) \tag{1}$$

with initial conditions,

$$u\left(x,y,0\right)=\varphi_0\left(x,y\right),\frac{\partial u(x,y,0)}{\partial t}=\varphi_1\left(x,y\right), \tag{2}$$

and boundary conditions

$$u\left(0,y,t\right)=u\left(a,y,t\right)=0,$$

$$u\left(x,0,t\right)=u\left(x,b,t\right)=0. \tag{3}$$

Note that equation (1) is the equation of vibrations of a plate-like structure. Here $\rho(x,y)$- plate density at a point (x,y), G- tension, $h(x,y)$- plate thickness, $u(x,y,t)$- plate displacement at point (x,y) in time t, $v\left(x,y,t\right)$- control function, $\varphi_0(x,y)$, $\varphi_1(x,y)$- given initial functions.

It is assumed that $(x,y)\in D=\{(x,y):0<x<a,\ 0<y<b\}$, $t\in(0,T)$. Let us denote $Q_T=D\times(0,T)$. Here a, b, T- given positive numbers.

For the class of admissible controls we take the set

$$U_{ad}=\left\{v\in L_2\left(D\right):\|\,v\|_{L_2(D)}\le M\right\}.$$

Here M- a given positive number.

The optimal control problem is set: to minimize the functional

$$J\left(v\right)=\frac{1}{2}\|u-u_0\|^2+\frac{a}{2}\|v\|^2_{L_2(D)} \tag{4}$$

in the U_{ad} class together with the solution of problem (1)-(3).

We will call this problem (1)-(4).

Note that similar problems are considered in [2], [3], [4], but in these works, control functions only appear in the coefficients of the derivatives with respect to spatial variables.

The following results were obtained in this paper:

Theorem 1. *Suppose that the above conditions are satisfied for problem* (1)-(3). *Then this problem has a unique generalized solution from the class* $u(x,y,t) \in W_2^1(Q_T)$.

Theorem 2. *Suppose that the above conditions are satisfied for problem* (1)-(4). *Then this problem has a unique optimal control.*

Finally, a necessary optimality condition is derived as an integral inequality.

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Integrable Systems with Dissipative Shift Generator*

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We present new examples of integrable dynamical systems of any odd order that are homogeneous in part of the variables. In these systems, subsystems on the tangent bundles of lower-dimensional manifolds can be distinguished. In the cases considered, the force field is partitioned into an internal (conservative) part and an external part. The external force introduced by a certain unimodular transformation has alternate dissipation; it is a generalization of fields examined earlier. Complete sets of first integrals and invariant differential forms are presented.

Keywords: dynamic system, integrability, dissipation, first integral with essential singular points, invariant differential form

As is known (see [1–3]), finding a sufficient number of tensor invariants (not only autonomous first integrals) facilitates the study and sometimes allows one to integrate a system of differential equations exactly. For example, the presence of an invariant differential form of the phase volume allows one to reduce the number of required first integrals. For conservative (in particular, Hamiltonian) systems, this fact is natural in the case where the phase flow preserves a volume with a smooth (or constant) density. The situation is more complicated (in terms of the smoothness of the invariants) for systems with attractive or repulsive limit sets. For

* The study was conducted under the state assignment of Lomonosov Moscow State University.

such systems, in general, the coefficients of the sought invariants should include functions with essentially singular points.

Our approach is that for exact integration of an autonomous system of order m, one must know $m-1$ independent tensor invariants. At the same time, to achieve exact integrability, one must also satisfy a number of additional conditions on these invariants.

Important cases of integrable systems with a finite number of degrees of freedom in a nonconservative force field have already been considered in the author's works [4–7]. The present study extends the results of these works to a wider class of dynamic systems. In these works, the emphasis was on finding a sufficient number of first integrals. However, as is known, sometimes a complete set of first integrals for systems may not exist, but a sufficient number of invariant forms can be provided.

For systems of classical mechanics, the concepts of "conservatism", "force field", "dissipation", etc. are quite natural. Since this paper studies dynamic systems on the tangent bundle to a smooth manifold (position space), we will clarify these concepts for such systems.

The analysis "in the whole" begins with the study of the given geodesic equations on an ndimensional surface, whose left-hand sides, with the correct parametrization, represent expressions of the coordinates of the acceleration of a material particle on such a surface, and the right-hand sides are equated to zero. Respectively, the quantities that are substituted into the right-hand side can be considered as some generalized forces. This approach is traditional for classical mechanics, and now it is naturally extended to a more general case of the tangent bundle of a smooth manifold. This allows one to construct "force fields". Thus, for example, by introducing into the system coefficients that are linear in one of the coordinates (in one of the quasi-velocities of the system) of the tangent space, we obtain a force field with dissipation of alternating sign (depending on the sign of the coefficient itself).

Although the phrase "dissipation of alternating sign" is somewhat contradictory, we will nevertheless use it. Taking into account that in mathematical physics, dissipation "with a 'plus' sign" is the dissipation of total energy in the usual sense, whereas dissipation "with a 'minus' sign" is a kind of "pumping up" of energy (in mechanics, the forces that ensure the dissipation of energy are called dissipative, and the forces that ensure the pumping up of energy are called accelerating).

The conservatism of systems on tangent bundles can be understood in the traditional sense, but we make the following remark. We say that a system is conservative if it has a complete set of smooth first integrals, which means that it does not have attractive or repulsive limit sets. If it does have such, then we say that the system in one or another domain of the phase space has dissipation of some sign and, as a consequence, the system has at least one first integral (if they exist at all) with essentially singular points.

In this activity, the force field is partitioned into the so-called internal and external parts. The internal field is characterized by the fact that it does not change the conservatism of the system, and the external field can introduce alternating dissipation into the system. We also note that the type of internal force fields is borrowed from the classical spatial dynamics of rigid bodies.

Thus, we present first integrals and invariant differential forms of classes of dynamic systems homogeneous in terms of variables of an arbitrary odd order $2n+1$, in which a system with n degrees of freedom on its $2n$-dimensional manifold can be distinguished. In this case, the force field is partitioned into an internal (conservative) field and an external field, which has the so-called alternating dissipation. The external field is introduced using a certain unimodular transformation and generalizes the force fields considered earlier.

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Mathematical Modeling of Credit Debt Repayment Strategies*

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We describe a mathematical model for the consumer debt repayment by a representative rational household. The model is formalized as an optimal control problem featuring a non-compact control set and an endogenous time horizon. We provide the analysis of the post-repayment utility functional and examine how the resulting expected profit of a commercial bank depends on the household’s chosen repayment strategy.

Keywords: mathematical modeling, optimal control, impulse control, maximum principle

1 The main results

We present a mathematical model for credit debt repayment by a representative rational household. The modeling of the economic behavior of households is based on the concept of a rational representative economic agent and arises to F. Ramsey. When issuing a consumer loan, a commercial bank relies on a stable flow of annuity payments to cover its own obligations.

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However, according to statistics, borrowers are prone to early repayment, which disrupts the expected cash flow at the present. In a high interest rate environment, the problem of estimating this cash flow is of particular importance.

The household borrows a consumer loan in order to increase its current consumption. After getting a consumer loan, the household chooses a debt repayment strategy that maximizes discounted consumption. To control the risk of default, the commercial bank sets the repayment time period and imposes restrictions on the repayment regime. We introduce the most common restriction that is the annuity payment. In case the household's repayment amount exceeds the annuity payment, the bank lowers the limit on the mandatory repayment without changing the repayment period.

The household's objective is to maximize a utility functional comprising two components: the first represents utility derived during the debt repayment period, and the second captures the expected utility from economic preferences after the loan is fully repaid. The model is formalized as an optimal control problem featuring a non-compact control set and an endogenous time horizon. The existence theorem for solutions in the class of functions admitting impulse control is proved. Necessary optimality conditions in the form of Pontryagin's maximum principle are investigated. We study the optimality conditions for the repayment strategy of the household. We obtain the optimality conditions for the early debt repayment regime, the annuity payment regime, and the combination of early repayment and the annuity payment regimes.

We provide an analysis of the post-repayment utility functional and examine how the resulting expected profit of a commercial bank depends on the household's chosen repayment strategy. To do this, we investigate a new model for the formation of interest rates on consumer loans based on an analysis of commercial interests and the logic of behavior of commercial banks. The model assumes that the borrowers' incomes are described by a geometric Brownian motion. The commercial banks assess the default risk of borrowers. According to the Feynman–Kac formula, this assessment is reduced to solving a boundary value problem for partial differential equations. An analytical solution to this problem is constructed. It is possible to reduce the solution of the boundary value problem to the Cauchy problem for the heat equation with an external source and obtain a risk assessment in analytical form with the help of the Abel equation.

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Detectability of Time-Varying Differential-Algebraic Equations with Scalar Output

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We consider nonstationary linear and nonlinear systems of differential-algebraic equations with a scalar observed output. In the linear case, sufficient detectability conditions are obtained and a state observer is constructed. For systems with a special type of controlled input, a stabilization procedure using dynamic output feedback is proposed. We obtain detectability conditions for a nonlinear systems by the linear approximation.

Keywords: differential-algebraic equations, state observer, stabilization, dynamic output feedback, detectability

1 Statement of the Problem

Consider the nonlinear differential-algebraic equations (DAEs)

$$f\left(t, x(t), \frac{d}{dt}x(t), u(t)\right) = 0, \tag{1}$$

$$y(t) = c^\top(t)x(t) + \rho(t, x(t)) \;\; t \in T = [0, +\infty), \tag{2}$$

where $x(t)$ is the desired vector function, $u(t)$ is a controlled input of the dimension l, $y(t)$ is the observed output, function $f(t,\alpha,\beta) : W \to \mathbf{R}^n$ is continuously differentiable with respect to each of their arguments a sufficient number of times in the domain $W = \{(t,\alpha,\beta) \in \mathbf{R}^{2n+1} : t \in T, \ \|\alpha\| < K_0, \ \|\beta\| < K_1\}$; $\rho(t,\alpha) : \hat{W} \to \mathbf{R}$, $\hat{W} = \{(t,\alpha) \in \mathbf{R}^{n+1} : \ t \in T, \ \|\alpha\| < K_0\}$, $c(t) : T \to \mathbf{R}^n$, $^\top$ stands for transposition.

It is assumed that

$$f(t,0,0) = 0, \quad \rho(t,0) = 0 \;\; \forall t \in T,$$

$$\det \frac{\partial f(t,\alpha,\beta)}{\partial \beta} = 0 \;\; \forall (t,\alpha,\beta) \in W.$$

Moreover, we analyse the linear system

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) + U(t)u(t) = 0, \tag{3}$$

$$y(t) = c(t)^\top x(t), \quad t \in T, \tag{4}$$

where $A(t)$ and $B(t)$ are given $(n \times n)$-matrices; $U(t)$ is known $(n \times l)$-matrix, $c(t)$ is a given n-dimensional vector function. $y(t)$ is a scalar observed output.

It is allowed the case

$$\det A(t) \equiv 0, \quad t \in T.$$

2 Main Results

For linear system (3), the conditions are formulated for the existence of the structural form with separated differential and algebraic subsystems, which is equivalent to the original system in the sense of solutions.

For this structural form with scalar output, a state observer is constructed, which can also be used as such for system (3), (4). In fact, to construct the observer, it is sufficient to construct an observer for the differential subsystem of DAEs (3).

We consider the linear system with a scalar control and the derivative of control. For such DAEs, the stabilizability conditions are obtained using a state observer. In this case, the stabilizing dynamic feedback on the output includes only the solutions of the differential subsystem of the observer.

When analyzing the nonlinear system (1) with scalar output (2), we used a structural form, any solution of which starting in a sufficiently small neighborhood of zero is a solution to system (1), and vice versa. This structural form is a nondegenerate system and is a part of the components of the implicit function satisfying the r-derivative array equations. For DAEs (1), (2), using a linear approximation system, we obtain sufficient conditions for detectability and construct a state observer. It should be noted that constructing the observer requires finding the implicit function mentioned above. However, verifying the detectability conditions does not require this, but relies exclusively on the properties of the linear approximation.

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Analitical Solution of the Dirichlet Problem for the Eikonal Equation in an Unbounded Domain

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Considers the n-dimensional eikonal equation, which describes the propagation of light in a homogeneous medium from a source on the boundary of a compact set

into an unbounded region of n-dimensional space. It is shown that the solution to this problem, obtained using the method of characteristics, can be represented analytically using a formula generalizing Kruzhkov's formula and is the entropy, viscosity, and minimax solution of the Dirichlet problem in the unbounded region.

Keywords: eikonal equation, homogeneous medium, unbounded region, method of characteristics, superdifferential

1 Introduction

Considers a mathematical model describing the propagation of light in a homogeneous physical medium from a source (the boundary of a compact set) into an unbounded region of n-dimensional space. It is shown that the solution to this problem, obtained via the generalized method of Cauchy characteristics, admits an analytical representation given by a formula generalizing the Kruzhkov formula[1].It is proved that this solution is superdifferentiable at all interior points of the considered unbounded domain, and that the superdifferential[2] coincides with the Clarke subdifferential[3]. This solution is the entropy solution[1], the minimax solution[2], and the viscosity solution[4] of the Dirichlet problem under consideration in an unbounded domain.

2 Problem statement

Consider the eikonal equation in $\mathbb{R}^n \ni x$ for a homogeneous medium ($n(x) \equiv 1$):

$$\sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 = n^2(x). \tag{1}$$

In $\mathbb{R}^n$, let $u_0(x)$ and $\phi(x)$ be twice continuously differentiable functions, and let the set $\Omega_0 = \{x : \phi(x) \leq 0\}$ be a compact subset of $\mathbb{R}^n$ with nonempty interior.

Consider the set $\Omega = \{x \in \mathbb{R}^n : \phi(x) \geq 0\}$, whose boundary is $\partial\Omega = \{x \in \mathbb{R}^n : \phi(x) = 0\}$.

Define in the space $\mathbb{R}^{n+1}$ the boundary manifold

$$C_0 = \{(x_0, u_0) \in \mathbb{R}^{n+1} : x_0 \in \partial\Omega, u_0 = u_0(x_0)\}. \tag{2}$$

Since the domain Ω is unbounded, the additionally condition at infinity is imposed:

$$u(x) \to -\infty \quad \text{as} \quad ||x|| \to \infty,\ x \in \Omega. \tag{3}$$

where $||\cdot||$ denotes the Euclidean norm.

The Dirichlet problem consists in constructing a function $u(x)$ that satisfies equation (1) in the domain Ω, its graph contains the manifold (2), the condition (3) is satisfied.

3 The main results

Let the set $K = \{y \in \mathbb{R}^{n-1} : x_0(y) \in \partial\Omega\}$ is compact.

For the equation

$$1 - \sum_{i=1}^{n} s_i^2 = 0$$

let's consider the solution of the characteristic system ODEs[5]

$$\begin{cases} x(t,y) = x_0(y) - 2ts_0(y), \\ s(t,y) = s_0(y), \\ z(t,y) = z_0(y) - 2t. \end{cases}$$

$$t \geq 0,\ x_0(y) = x_0 \in \partial\Omega,\ z_0(y) = u_0(x_0(y)), y \in K.$$

$$||s_0(y)||^2 = 1,\ \ \sum_{i=1}^{n} s_{0_i}(y) \cdot \frac{\partial x_{0_i}(y)}{\partial y_j} = \frac{\partial z_0(y)}{\partial y_j}, j = 1, \cdots n-1, y \in K$$

A characteristic $x(t, y^*)$ is called admissible if $y^* \in K$ and $x(t, y^*) \in \Omega$ for all $t \geq 0$. For admissible characteristics, we define a function

$$u(x) = \max_{y^*:\, x(t,y^*)=x} \{z(t, y^*)\},\ x \in \Omega. \tag{4}$$

Certain properties of this function are proved.

Theorem 1. *The function $u(x)$ of the form* (4) *coincides with the function $v(x)$ of the form*

$$v(x) = \max_{y \in K} z_0(y) - ||x_0(y) - x||, \quad x \in \Omega. \tag{5}$$

Note that the formula (5) coincides with the Kruzhkov formula for the entropy solution of the eikonal equation for the Dirichlet problem inside a compact set Ω.

Theorem 2. *The function $u(x)$ of the form* (4) *is continuous and locally Lipschitz on Ω. The superdifferential $D^+u(x)$[2] is nonempty at every point $x \in \Omega\ \partial\Omega$, and it coincides with the Clarke subdifferential[3]. The function $u(x)$ belongs to the stability class $E(\Omega)$[1] and it is the entropy solution of the Dirichlet problem* (1) - (3).

Theorem 3. *The function $u(x)$ of the form* (4) *coincides with the minimax[2] and the viscosity solution[4] of the problem* (1) - (3).

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Retinal Image Restoration Using Total Variation with Overlapping Group Sparsity*

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Keywords: image restoration, retinal imaging, total variation, overlapping group sparsity, ADMM

1 The main results

In this work, we consider the problem of adaptive optics (AO) retinal image restoration. This problem involves an only partially known point spread function (PSF) [1]. The characteristics of AO retinal images determine that their global PSF can be represented by combining unknown parameters with a few PSFs, thus requiring the simultaneous estimation of both the image and the parameters. Total variation (TV) regularization is an efficient regularization technique for AO retinal image restoration. However, a main shortcoming of TV regularization is its potential to experience the staircase effects, particularly in smooth regions of the image. To overcome this limitation, we propose a myopic image deconvolution model incorporating overlapping group sparse total variation (OGSTV) as the regularization term. The proposed model is formulated as follows:

$$\begin{cases} \min\limits_{x\in\mathcal{C}_x, w\in\mathcal{C}_w} & \Phi\left(x,w\right)=\frac{\mu}{2}\|A\left(w\right)x-d\|_2^2+\varphi(\nabla x) \\ \text{s.t.} & \sum\limits_{j=1}^{p} w_j = 1, \end{cases} \tag{1}$$

where $x\in\mathbb{R}^{n^2}$ represents unknown real images, $w\in\mathbb{R}^p$ is an unknown parameter. $\mathcal{C}_x=\{x \mid x_i\geq 0 \text{ for } i=1,2,\dots,n^2\}$, $\mathcal{C}_w=\{w \mid w_j\geq 0 \text{ for } j=1,2,\dots,p\}$. $\frac{\mu}{2}\|A\left(w\right)x-d\|_2^2$ denotes the data fidelity term, $A(w)=\sum_{j=1}^{p} w_j A_j$ is the blurring matrix defined by PSF, $\varphi(\nabla x)$ denotes the OGSTV regularization function, and $\mu>0$ denotes the weighting parameter to balance the two terms in the objective function.

To numerically solve model (1), we introduce artificial variables to rewrite the problem in an equivalent form, and employ the alternating direction method of multipliers (ADMM) as the outer-layer optimization framework, yielding the following iterative scheme:

$$\begin{cases} v^{k+1} = \arg\min\limits_{v} L(x^k, w^k, v, \alpha^k), \\ (x^{k+1}, w^{k+1}) = \arg\min\limits_{x,w} L(x, w, v^{k+1}, \alpha^k), \\ \alpha^{k+1} = \alpha^k + \rho\left(\nabla x^{k+1} - v^{k+1}\right), \end{cases}$$

where $L(x,w,v,\alpha)$ is the augmented Lagrangian function.

For the v-subproblem, we employ the majorization-minimization (MM) method [2] to address the problem involving the OGSTV term. For the tightly coupled (x,w)-subproblem,

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the linearize and project (LAP) method [3] is applied. Then, we propose the ADMM-MM-LAP method. Consequently, numerical results demonstrate that the proposed OGSTV model significantly outperforms the existing state-of-the-art TV model on multiple evaluation metrics.

We employ the following example to show the numerical results. All computational results were obtained using MATLAB 2020a on a PC with 16.0 GB RAM and AMD Ryzen 5 4600U with Radeon Graphics at 2.10 GHz.

Example: The test image is a 256×256 segment extracted from an adaptive optics (AO) retinal image. We construct the test problem using the IR Tools regularization toolbox, which generates blur at three distinct levels ('mild', 'medium', 'severe'). This toolbox simulates spatially invariant Gaussian blur through its $PRblurgauss$ function and out-of-focus blur via the $PRblurdefocus$ function. In this example, we build a combined PSF using $PRblurgauss$ with a 'mild' BlurLevel and $PRblurdefocus$ with a 'mild' BlurLevel. The relative errors of x and w are denoted by $\mathrm{RE_x}$ and RE_w. $\mathrm{RE_x}=\frac{\|x-x^*\|_2}{\|x^*\|_2}$, where x^* is the true image and x is the restored image. $\mathrm{RE}_w=\frac{\|w-w^*\|_2}{\|w^*\|_2}$, where w^* is the real parameter and w is the calculated parameter. Figure 1 plots the relative errors of both the reconstructed image and the estimated parameters against iteration.

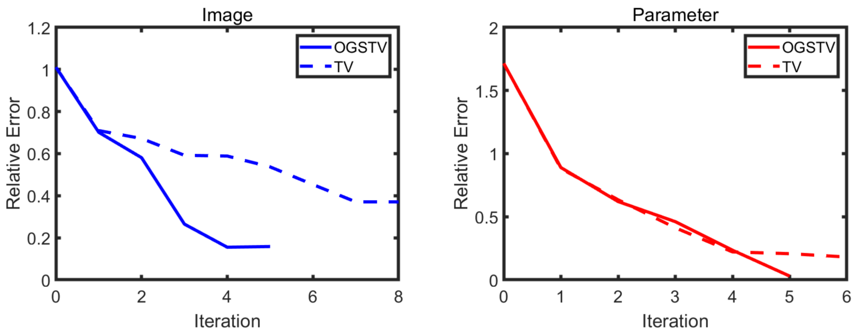

Figure 1: $\mathrm{RE_x}$ (left) and $\mathrm{RE_w}$ (right)versus iteration number

It can be seen from Figure 1 that the relative error curves of the proposed OGSTV model decline more steeply. This indicates that the OGSTV model achieves both more efficient and accurate image reconstruction and parameter estimation, as well as higher SNR, PSNR, and SSIM values.

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Analysis of Fixed Points and Bifurcations in a Generalized Lorenz System with Cross Nonlinearities

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This paper considers a generalized Lorenz system containing cross nonlinear terms with coefficients C_1, C_2, C_3 satisfying $C_1 + C_2 + C_3 = 0$. For this system, global solvability of solutions is proved and dissipativity is established using the guiding function method. A complete analytical analysis of fixed points depending on the system parameters is carried out. For the particular cases $C_1 = 0$ and $C_2 = 0$, the characteristic equations corresponding to nonzero fixed points are obtained and theorems on the distribution of roots describing the Andronov–Hopf bifurcation are proved. In the case $C_1C_2C_3 \neq 0$, conditions for the existence of two or four nonzero fixed points are identified.

Keywords: generalized Lorenz system, fixed points, bifurcations, dissipativity, guiding function method, Andronov–Hopf bifurcation

1 The main results

We study a system of differential equations generalizing the classical Lorenz model:

$$\begin{cases} x_1' = -\sigma x_1 + \sigma x_2 + C_1 x_2 x_3 \\ x_2' = r x_1 - x_2 + C_2 x_1 x_3 \\ x_3' = -b x_3 + C_3 x_1 x_2 \end{cases} \tag{1}$$

where $\sigma > 0$, $b > 0$, $r > 0$, the coefficients C_1, C_2, C_3 satisfy $C_1 + C_2 + C_3 = 0$ and $|C_1| + |C_2| + |C_3| > 0$. The classical Lorenz system is obtained as a special case when $C_1 = 0$, $C_2 = -1$, $C_3 = 1$. The presence of cross nonlinearities enriches the dynamical behavior of the system.

It is proven that any solution of the system (1) is defined for all $t \in (-\infty, +\infty)$. In the case when $C_3 \neq 0$, dissipativity is established using a guiding function:

$$V(x_1, x_2, x_3) = x_1^2 + x_2^2 + \left(x_3 - \frac{\sigma + r}{C_3} \right)^2, \tag{2}$$

which guarantees the existence of a globally attracting bounded set in the phase space. In the case $C_3 = 0$, dissipativity is equivalent to the condition $r < 1$, which corresponds to the stability of the zero equilibrium.

Special cases where $C_1C_2C_3 = 0$ are considered, stationary points are found, and some of their properties are established. When $C_1 = 0$ and $r > 1$, the system of equations (1) has two nonzero stationary points:

$$O^{\pm} = \Big(\pm\sqrt{b(r-1)/C_2^2}, \pm\sqrt{b(r-1)/C_2^2}, (1-r)/C_2 \Big).$$

At these points, the Jacobian matrix of the system (1) corresponds to the same characteristic equation:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(r + \sigma)\lambda + 2\sigma b(r - 1) = 0. \tag{3}$$

On the properties of the roots, the following holds

Theorem 1. *Let $r > 1$ and let $\lambda_i, i = 1, 2, 3,$ be the roots of equation* (3)*. Then:*

1. If $\sigma \le b + 1$, then $Re\lambda_i < 0$ for all $i = 1, 2, 3$.
2. If $\sigma > b + 1$, then there exist numbers r_0 and r_1 such that

$$1 < r_0 < r_1,$$

$$r_1 = \sigma(\sigma + b + 3)/(\sigma - b - 1)$$

and 1) for $1 < r < r_0$, all roots $\lambda_i, i = 1, 2, 3$ are real and negative;
2) for $r_0 < r < r_1$, one root, say λ_1 is real and negative, and the other two are complex conjugates with negative real part: $\lambda_{2,3} = \alpha \pm i\beta$, where $\alpha < 0, \beta > 0$;
3) for $r > r_1, \lambda_1$ is real, and $\lambda_{2,3} = \alpha \pm i\beta$, where $\alpha > 0, \beta > 0$.

In the case when $C_2 = 0$ and $r > 1$, the system of equations (1) has two nonzero stationary points:

$$O^{\pm} = \Big(\pm 1/r\sqrt{b\sigma(r-1)/C_1^2}, \pm\sqrt{b\sigma(r-1)/C_1^2}, \sigma(1-r)/C_1 r\Big).$$

At these stationary points, the Jacobian matrix of the right-hand sides of the system of equations (1) has the following characteristic equation:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(1 + \sigma r)\lambda + 2\sigma b(r - 1) = 0. \tag{4}$$

For the roots of equation (4), a similar theorem can be formulated.

In the case $C_3 = 0$ and $r \neq 1$ the system has only the zero stationary point. If $C_3 = 0$ and $r = 1$ then all points of the form $(a, a, 0)$, where $a \in \mathbb{R}$, are stationary.

In the case where $C_1 C_2 C_3 \neq 0$, the system of equations (1) may have 0, 2, or 4 nonzero stationary points. For $r > 1$, there are exactly 2 nonzero stationary points. For $r < 1$ and under the following conditions

$$(C_1 r + C_2\sigma)^2 - 4C_1C_2\sigma(r-1) > 0, \quad C_1 r + C_2\sigma \neq 0, \quad C_1C_2 < 0 \tag{5}$$

the system has four nonzero stationary points, which fundamentally distinguishes this generalization from the classical Lorenz model. If $r < 1$ and at least one of the conditions in (5) is violated, the system of equations (1) has no nonzero stationary points.

The presented results are new and develop analytical approaches to the study of generalized Lorenz systems, characteristic of Russian scientific schools [1,2,3]. In contrast to the majority of works that employ numerical methods, this research provides rigorous analytical criteria for the existence and stability of stationary points. The obtained results lay the foundation for further analysis of bifurcations leading to the birth of limit cycles, investigation of homoclinic structures [1], and the search for hidden attractors in the considered class of systems.

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Construction of Neural Differential Equations by Given Trajectory Points

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We develop an approach to building neural ordinary differential equations (ODEs) using methods of inverse problems of dynamics. We obtain a forward-time ODE system and an extended reverse-time ODE system for training a neural network that produces neural ODEs from a given set of trajectory points.

Keywords: neural ODE, inverse problem of dynamics, gradient descent

Inverse problems of dynamics [1] involve analytical construction of equations of motion with given properties. The primary tool for solving inverse problems of dynamics is the method of constructing an ODE system from a given integral curve [2]. Moreover, the equations of motion can be built in such a way that the motion along a given integral curve is stable [3].

Neural ODEs [4] also address the problem of building equations of motion with given properties. However, the functions on the right-hand sides of neural ODEs are not expressed analytically, but are instead the output of a neural network. The neural network is trained using L. S. Pontryagin's approach [5] by numerically solving an extended ODE system in reverse time. The difficulties with neural ODEs include both instability of the neural network training and instability of trajectories of the neural ODEs. Therefore, we will develop an approach to building and training neural ODEs based on the method of constructing an ODE system from a given curve [2].

We study an ODE system

$$\dot{\mathbf{x}} = f\left(x, t, \theta\right), x = \left(x_1, x_2\right)^T, x\left(t_0\right) = x_0 \in \mathbb{R}^2, t \in \left[t_0, t_N\right], \tag{1}$$

where $f\left(x,t,\theta\right) = \left(f_1, f_2\right)^T = \left(-\alpha\omega\frac{\partial\omega}{\partial x_1} + \Psi\frac{\partial\omega}{\partial x_2}, -\alpha\omega\frac{\partial\omega}{\partial x_2} - \Psi\frac{\partial\omega}{\partial x_1}\right)^T$, scalar functions $\omega\left(x,\theta\right)$, $\Psi\left(x,\theta\right)$ are produced as the output $\left(\omega,\Psi\right)$ of a neural network $\mathcal{N}$ with the input $x \in \mathbb{R}^2$ and parameters $\theta \in \mathbb{R}^p$.

The system (1) admits an integral curve implicitly determined by algebraic equation [2]

$$\omega\left(x,\theta\right) = 0, \left\|\nabla_x\omega\right\|^2 = \left(\frac{\partial\omega}{\partial x_1}\right)^2 + \left(\frac{\partial\omega}{\partial x_2}\right)^2 \neq 0, \tag{2}$$

and for $\alpha > 0$ the motion along the curve (2) is stable [3].

We assume that the data set $\widetilde{D} = \{(\tilde{x}_k, t_k)\}_{k=1}^{N}$ of trajectory points $\tilde{x}_k = x(t_k)$ of system (1) at times $t_k, k = \overline{1, N}$, such that $t_0 < t_1 < ... < t_k < ... < t_N$, is given. We study the problem of building and training a neural network $\mathcal{N}$ with the output $(\omega(x;\theta), \Psi(x;\theta))$ for system (1) based on the dataset $\widetilde{D}$.

We assume that the deviation of the trajectory $x(t;\theta)$ of system (1) from a given point $\tilde{x}_k$ at time t_k is estimated by a smooth scalar function $\rho(x(t_k;\theta), \tilde{x}_k)$, and the loss functional of the neural network $\mathcal{N}$ is the sum of deviations of the trajectory $x(t;\theta)$ from the points of the data set $\widetilde{D}$:

$$\mathcal{L}[x(t;\theta)] = \sum_{k=1}^{N} \rho(x(t_k;\theta), \tilde{x}_k). \tag{3}$$

Training of the neural network $\mathcal{N}$ with gradient descent methods [6] requires calculation of the gradient of the loss functional $\mathcal{L}$ with respect to the neural network parameters θ. Following [4], we introduce N extended ODE systems in reverse time for $t \in [t_k, t_{k+1})$, $k = N-1, N-2, \dots, 0$ with terminal conditions to calculate $\nabla_\theta \mathcal{L}$:

$$\begin{cases} \dot{x} = f(x,t,\theta), & x(t_{k+1}) = x_{k+1}, \\ \dot{\chi} = -\nabla_x \left[f^T(x,t,\theta)\chi\right], & \chi(t_{k+1}) = \nabla_x \rho(x_{k+1}, \tilde{x}_{k+1}), \\ \dot{\vartheta} = -\nabla_\theta \left[f^T(x,t,\theta)\chi\right], & \vartheta(t_{k+1}) = 0. \end{cases} \tag{4}$$

We solve system (4) in reverse time for $t \in [t_k, t_{k+1})$ and obtain $\vartheta(t_k;\theta)$, which is a term in $\nabla_\theta \mathcal{L}$. Summing up all the terms $\vartheta(t_k;\theta)$, we get the gradient of the loss functional $\mathcal{L}$:

$$\nabla_\theta \mathcal{L}[x(t;\theta)] = \sum_{k=0}^{N-1} \vartheta(t_k;\theta). \tag{5}$$

The algorithm for training the neural network $\mathcal{N}$ with the obtained gradient $\nabla_\theta \mathcal{L}[x(t;\theta)]$ is similar to the training algorithm in [7].

Thus, the construction of the ODE system (1) is based on the method by N.P. Erugin [2], and training of the neural network is based on solving the extended ODE system (4) in reverse time according to the ideas of L.S. Pontryagin [5].

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Neural-Enhanced Stochastic Inverse Control

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Inverse optimal control represents one of the most challenging problems in control theory, where we seek to infer cost functions from observed optimal trajectories rather than designing controllers for predetermined objectives. This paper presents a comprehensive survey of neural-enhanced stochastic inverse control methods, examining how deep learning architectures have revolutionized our ability to solve these traditionally intractable problems. We trace the evolution from classical variational approaches through modern data-driven techniques, highlighting the critical role of stochastic approximation in handling real-world uncertainties. The survey synthesizes recent advances in neural network architectures designed explicitly for inverse problems, including physics-informed networks, attention mechanisms, and generative models. Through systematic analysis, we identify convergence patterns, computational trade-offs, and theoretical guarantees that define the current landscape. Our comparative framework reveals that while neural approaches achieve unprecedented empirical success in high-dimensional settings, significant gaps remain between practical performance and theoretical understanding. We conclude by outlining seven open problems that could shape the next generation of inverse control algorithms, particularly in safety-critical applications where probabilistic guarantees are essential.

Keywords: Inverse optimal control, stochastic differential equations, neural networks, optimization, reinforcement learning

High-Order Robust Numerical Differentiation in Motion Determination Problems

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A robust method for high-order numerical differentiation is proposed that can be used to estimate kinematic motion parameters from noisy measurements. The method is based on state-space modeling with a nilpotent matrix and two-stage variable scaling, ensuring independence from the sampling step. Increased state vector dimension is shown to improve accuracy, but computational error begins to dominate for $n \geq 7$. The developed approach is suitable for both post-processing and real-time onboard applications.

Keywords: numerical differentiation, ill-posed problems, kinematic motion parameters, variable scaling

This paper addresses a central problem of measurement processing — numerical multiple differentiation of an arbitrary measurable function $x(t)$ under finite measurement and computation accuracy, focusing on high-precision estimation of kinematic motion parameters as a reference result. The classical inverse problem of differentiation is ill-posed (in the sense of Hadamard) — small errors in the input data can lead to a catastrophic increase in errors in derivative estimates. The relevance of this problem is well supported by fundamental applied tasks in navigation, where the key information consists of kinematic motion parameters — velocity, acceleration, jerk — while, as a rule, only integral characteristics such as coordinates, ranges, signal phases, and ground speed are accessible with high precision. Generalizing this formulation leads to the well-known inverse problem of Newtonian mechanics — determining the force (cause) of a known (or observed) trajectory [1]. Moreover, having a high-quality solution to the basic problem makes it possible to consider problems in gravimetry and mobile gravimetry [2,3].

The main methodological achievement of this work is the development of a robust computational procedure for numerical differentiation [4,5]. The evolution model of the measured function $x(t)$ is constructed in a state space with a nilpotent matrix $\mathbf{A}$ ($\mathbf{A}^n = 0$). On the discretization interval τ, a transition matrix is formed, which has an upper triangular form. Then, two-stage variable scaling is performed:

1. Transition to normalized variables $\mathbf{x}_k = \mathbf{D}_f \mathbf{f}_k$ using the diagonal matrix

$$\mathbf{D}_f = \mathrm{diag}\left(1, \tau, \frac{\tau^2}{2!}, \frac{\tau^3}{3!}, \dots, \frac{\tau^{n-1}}{(n-1)!}\right),$$

   as a result of which the new matrix $_f$ and the observability matrix $\mathbf{H}_f$ become integer-valued and independent of the sampling step τ.
2. If necessary, a second scaling $\mathbf{f}_k = \mathbf{D}_s \mathbf{s}_k$ is performed based on the column norms of the observability matrix $\mathbf{N}_f$.

The proposed hypothesis is that expanding the spectrum (dimension of) of the vector $\mathbf{x}(t)$ contributes to improving the accuracy of its component estimates. The validity of the hypothesis is confirmed by a demonstration example (fig. 1), which considers a certain measurable function $x(t) = e^{at}$ under double-precision computations ($\varepsilon_1 = 2{,}2 \cdot 10^{-16}$) and in the absence of measurement errors ($\varepsilon = 0$). Differentiation was performed for different n (dimension of state vector).

It can be seen from the example that for $n \geq 7$, the accumulation of computational error begins to dominate. This effect can be shifted to larger n by using quadruple precision.

The organization and quality of problem solving are directly related to their target formulation. Fundamental multi-purpose research requires large information and time resources, for which post-processing is advisable, and its effect can be significantly enhanced using wavelet technology [6]. For applied navigation tasks, the solution must be performed in real time onboard. The developed methodology is a step in both directions, providing robustness, covariance, and computational efficiency [5].

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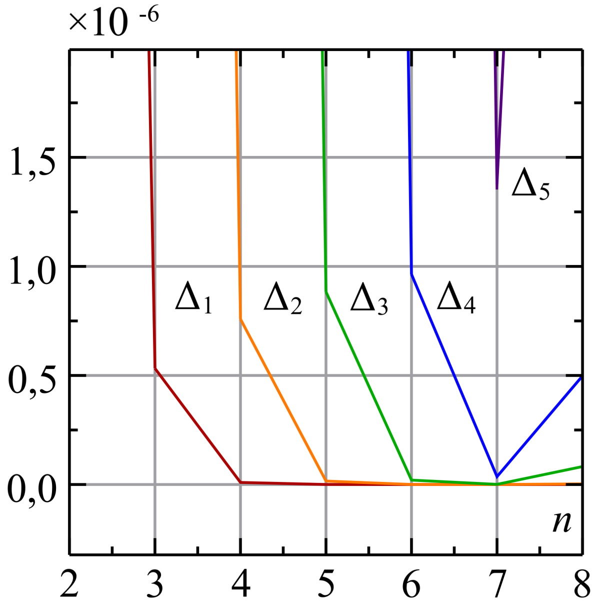

Figure 1: Graph of error magnitudes for estimates of the exponential function and its derivatives for various state vector dimensions.

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Indirect Algorithms in Optimal Control of Diffusion Processes*

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The talk is devoted to indirect algorithms for optimal control problems governed by diffusion processes. The stochastic problem is rewritten as a deterministic control problem for the law of the process, which satisfies the Fokker–Planck–Kolmogorov equation. This representation makes it possible to combine a forward equation for

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the probability density with a backward Kolmogorov equation for the transported terminal observable. The resulting exact increment formula gives not only an optimality condition, but also a constructive rule for improving the current control. The method is aimed at objectives depending nonlinearly on expectations and at applications where one needs implementable feedback laws under uncertainty, in particular in energy-storage management.

Keywords: diffusion processes, stochastic optimal control, Fokker–Planck–Kolmogorov equation, backward Kolmogorov equation, indirect methods

We consider controlled diffusion processes in $\mathbb{R}^n$,

$$\mathrm{d}X_t = f_t[X,u](X_t)\,\mathrm{d}t + G(X_t)\,\mathrm{d}W_t,$$

with open-loop or Markovian controls. The cost may depend on the distribution of X_t; for instance, it may contain variance terms, moment penalties, or nonlinear expressions of the form $\Psi(\mathbb{E}\ell(X_T))$. Such criteria arise when the aim is not to optimize one realization, but to regulate the behaviour of an ensemble. This is typical for stochastic models of storage systems, smart grids, and other engineering systems operating under random demand and production.

Let μ_t be the law of X_t. Under standard assumptions, μ_t solves the Fokker–Planck–Kolmogorov equation

$$\partial_t \mu_t = L^*_{u_t}\mu_t, \quad L_{u_t}\phi = D^{ij}\partial^2_{ij}\phi + f^i_{u_t}\partial_i\phi, \quad D = \frac{1}{2}GG^{\mathrm{T}}.$$

Thus the original stochastic problem can be treated as a deterministic optimization problem on a space of probability measures. For a fixed reference control $\bar{u}$ one then solves the backward Kolmogorov equation

$$\partial_t \bar{p}_t + L_{\bar{u}_t}\bar{p}_t = 0, \quad \bar{p}_T = \ell.$$

The function $\bar{p}_t$ is the terminal observable transported backwards along the diffusion generated by $\bar{u}$. It plays the role of an adjoint variable, but remains a function on the state space.

The main computational step is based on an exact increment formula. If u is another admissible control, then the difference between the two costs can be written through the difference of the corresponding Hamiltonian densities. In the affine case the characteristic term has the form

$$\Psi'(\langle \mu^u_t, \bar{p}_t\rangle)\langle \mu^u_t, \nabla\bar{p}_t \cdot (f_{u_t} - f_{\bar{u}_t})\rangle,$$

with an additional quadratic contribution when the energy of the control is penalized. This identity is not a linearization: it compares the reference control with a new admissible control directly. Therefore it can be used as a descent mechanism rather than only as a necessary condition.

The resulting algorithm has the usual indirect structure. Starting from u^k, one propagates the law forward, solves the backward Kolmogorov equation, and chooses u^{k+1} by minimizing the Hamiltonian expression obtained from the increment formula. For quadratic control costs this minimization is explicit. Depending on the admissible class, it gives a Markovian feedback, a density-feedback law, or a broadcast law depending only on averages over the current law. If the minimization step is exact, the cost decreases monotonically, and no separate line search is needed.

This construction is close in spirit to positional improvement methods. The control is recomputed from the current distribution and from the backward observable, so the method is nonlocal in the law even when the original diffusion has local coefficients. Numerically, the forward and backward equations may be solved on grids in low dimensions; in higher dimensions the same scheme can be combined with Monte Carlo simulation, kernel density approximation, or parametric approximations of the backward solution.

In the smart-grid example, the state includes the battery charge and the power exchanged with the external grid. The diffusion term reflects uncertainty of demand and renewable generation. The cost penalizes deviation of the charge from a useful middle level, variability of the grid load, and excessive control effort. The indirect algorithm then produces implementable charging and discharging laws which use both the current probability distribution and the adjoint information obtained from the backward Kolmogorov equation.

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Semiclassical Asymptotics of the One-dimensional Nonlocal Fisher-KPP Equation with a Fractal Time Derivative in the Weak Diffusion Approximation

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We use an approach for constructing quasiparticle-like asymptotic solutions within the weak diffusion approximation for the generalized population Fisher–Kolmogorov–Petrovskii–Piskunov (Fisher–KPP) equation, which incorporates nonlocal quadratic competitive losses and a fractal time derivative of non-integer order α (where $0 < \alpha \leq 1$). For the obtained solutions, the influence of the fractal parameter α on the behavior of quasiparticles is investigated.

Keywords: fractal calculus, nonlocal Fisher-KPP equation, semiclassical asymptotics, Maslov method, quasiparticles

The method of semiclassical approach based on Maslov's germ theory, originally developed in quantum theory, has found broad applications in a range of linear and nonlinear mathematical physics problems. In this paper, we extend the formalism of the method of semiclassical asymptotics to the nonlocal generalized population Fisher–KPP equation [2] with a fractal time derivative ($D^{\alpha}_{F,t}$) of order α [3]. Within the framework of the weak diffusion approximation, asymptotic solutions are derived that describe the fractal dynamics of quasiparticles. The fractal time derivative is introduced within the framework of the F_α-calculus [4,5]. The Fisher–KPP equation is decomposed into a system of nonlinear equations describing the dynamics of interacting quasiparticles in the class of trajectory-concentrated functions. The key element in constructing approximate quasiparticle solutions is the interaction between the dynamic system of quasiparticle moments (the Einstein-Ehrenfest system) and an auxiliary linear system of equations that is related to the original nonlinear system. The constructed asymptotic solutions demonstrate a non-trivial geometric structure. Each solution involves a finite number of localized distributions (quasiparticles), with each quasiparticle concentrated in a certain spatial neighborhood that determines its coordinates.

The paper considers an example illustrating the influence of fractality on physical diffusion processes in population dynamics using the example of a solution that includes two quasiparticles. Each quasiparticle initially follows a Gaussian density distribution. The obtained partial solutions show that diffusion controlled by the fractal time derivative ($D^\alpha_{F,t}$) with $\alpha < 1$ proceeds faster than diffusion described by the standard first-order derivative ($\alpha = 1$). This acceleration is due to the nature of the underlying time intervals: while ordinary diffusion evolves continuously in time, fractal diffusion $\alpha < 1$ proceeds in jumps, with events occurring at discrete points within the fractal Cantor time set.

The results presented here provide a foundation for developing approximate methods for the nonlocal Fisher–KPP equation with fractal spatial partial derivatives, a topic of particular interest in both the theory of fractal equations and their applications in biology and physics [6].

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Differential Equations of Neutral Type with Distributed Delay and a Model of Motion of an Elastic Aircraft*

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We consider a system of differential equations of neutral type with distributed delay, in which the derivative of an unknown vector-function with respect to the delayed argument is contained in the integral term. Estimates for the norm of solutions to the system are obtained, from which the exponential stability of the zero solution follows. The rate of exponential decrease of solutions at infinity is indicated. The obtained results are used to study the system of equations of motion of an elastic aircraft.

Keywords: equations of neutral type, distributed delay, exponential stability, estimates for solutions, Lyapunov–Krasovskii functional, elastic aircraft

We consider a system of differential equations of neutral type with distributed delay of the following form $\dfrac{d}{dt}y(t) = Ay(t) + \displaystyle\int_{-\infty}^{t} D(t-s)\dfrac{d}{ds}y(s)ds, \quad t > 0,$ where A is a constant matrix, $D(\xi)$ is a variable matrix with continuous elements, for which the estimate holds $\|D(\xi)\| \leq Me^{-\varepsilon\xi}, \quad \xi > 0,$ with constants $M > 0$ and $\varepsilon > 0$.

The exponential stability of the zero solution to the considered system is investigated. We obtain conditions for matrices A and $D(\xi)$, under which for the solution $y(t)$ the estimate of the following form is valid

$$\|y(t)\| \leq ce^{-\alpha t}, \quad t > 0,$$

where $c, \alpha > 0$. The constants c and α are written out explicitly.

When obtaining estimates for solutions, a method based on the construction of Lyapunov–Krasovskii functionals of a special type is used (see, for example, [1]).

The obtained results are used to study the system of equations of motion of an elastic aircraft, in which elastic deformations and aerodynamic effects are taken into account with linear accuracy relative to the kinematic parameters characterizing the motion and deformation of the aircraft [2].

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Inverse Problem for the Time-Fractional Linearized Kuramoto-Sivashinsky Equation in the Class of Decreasing Functions

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We consider an inverse problem associated with the linearized time-fractional Kuramoto-Sivashinsky equation in the Schwartz space, which is a complete topological space generated by a family of semi-norms. The inverse problem is formulated as the initial condition and integral overdetermination conditions.

Keywords: time-fractional Kuramoto-Sivashinsky equation, Schwartz space, Caputo fractional derivative, existence and uniqueness

The Kuramoto-Sivashinsky equation is a familiar nonlinear model arising in fluid dynamics, plasma physics, and reaction-diffusion systems. Implies a highly developed such as instabilities, turbulence, and pattern formation. Inverse problem for partial differential equations play an important role in many areas of applications and to create again unexplored forcing terms from additional experimental data. This paper examines the study of an inverse source problem for the following fractional differential equation.

We define the set of continuous functions $SC(T) = C([0,T]; S(\mathbb{R}))$, where $S(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions in the variable x. Following [1], we introduce the topological space $SC_\beta(T) = \{u \in SC(T) : \partial_{0t}^{\beta} u \in SC(T)\}$.

We constructed the linear part of the Kuramoto- Sivashinsky equation equipped with a Caputo fractional time derivative:

$$\partial_{0t}^{\beta} u + a^2 u_{xxxx} + b u_{xxx} + c u_{xx} + d u_x + ku = f(t)g(x,t) + h(x,t), \quad x \in \mathbb{R}, \quad 0 < t \le T, \tag{1}$$

with initial condition

$$u(x,0) = \varphi(x), \quad x \in \mathbb{R}, \quad 0 < t \le T, \tag{2}$$

and the overdetermination condition

$$\int\limits_{-\infty}^{+\infty} \eta(x) u(x,t) dx = \psi(t), \quad 0 \le t \le T, \tag{3}$$

where $0 < \beta < 1$, $a > 0$, b, c, d, and k are real numbers, $\eta(x)$ and $\psi(t)$ are known functions, $\varphi(x) \in S(\mathbb{R})$, $f(t) \in C[0,T]$. We obtained an exact analytical solution of the direct problem by employing the Fourier transform method and properties of the Caputo fractional derivative ([2], [3]). We tackle the inverse problem by transforming it into the problem of solving an operator equation [4]. Then we came to the conclusion that the resolvent operator is appropriately defined in the inverse problem, as demonstrated by the overdetermition condition.

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On Cubature Formulas for Multidimensional Singular Integrals with a Cauchy Kernel

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By approximating the density of a multidimensional singular integral with a Cauchy kernel using a Hermite-Fejér polynomial with Chebyshev nodes of the first kind, a cubature formula in the class of Hölder functions is constructed and analyzed.

Keywords: singular integral, interpolation, cubature formula

Consider a singular p-dimensional integral understood in the sense of the Cauchy principal value:

$$If = I(f; x_1, \dots, x_p) = \int_{-1}^{1} \dots \int_{-1}^{1} \frac{f(t_1, \dots, t_p)}{(t_1 - x_1) \cdot \dots \cdot (t_p - x_p)} dt_1 \dots dt_p,$$
$$-1 < x_j < 1,\ j = \overline{1, p}, \tag{1}$$

where $f(x_1, \dots, x_p)$ is the given density of the integral.

We denote by $H_{n_1,\dots,n_p}(f; x_1, \dots, x_p)$ the Hermite-Fejér polynomial (see, e.g., [1]) satisfying the conditions:

$$H_{n_1,\dots,n_p}(f; x_{j_1}^{(n_1)}, \dots, x_{j_p}^{(n_p)}) = f(x_{j_1}^{(n_1)}, \dots, x_{j_p}^{(n_p)}),$$
$$\left(\frac{\partial}{\partial x_k} H_{n_1,\dots,n_p}(f; x_1, \dots, x_p) \right)_{x_k = x_{j_k}^{(n_k)}} = 0, \quad j = \overline{1, n_k},\ k = \overline{1, p},$$

where $x_{j_k}^{(n_k)} = \cos \frac{2j_k - 1}{2n_k} \pi$ $(k = \overline{1, p})$ are the zeros of the Chebyshev polynomials of the first kind $T_{n_k}(x_k) = \cos(n_k \arccos x_k),\ k = \overline{1, p}$.

Approximating the density of integral (1) by a polynomial $H_{n_1,\dots,n_p}(f; x_1, \dots, x_p)$, we obtain the cubature formula

$$If = \sum_{h_1=1}^{n_1} \dots \sum_{h_p=1}^{n_p} f(x_{h_1}^{(n_1)}, \dots, x_{h_p}^{(n_p)}) \widetilde{A}_{h_1}^{(n_1)}(x_1) \cdot \dots \cdot \widetilde{A}_{h_p}^{(n_p)}(x_p) + R_{n_1,\dots,n_p} f, \tag{2}$$

where $R_{n_1,\dots,n_p} f = R_{n_1,\dots,n_p}(f; x_1, \dots, x_p)$ is the remainder term,

$$\begin{aligned}\widetilde{A}_{h_i}^{(n_i)}(x_i) =& \frac{1}{2n_i^2} \left[\frac{1 - x_i x_{h_i}^{(n_i)}}{\left(x_i - x_{h_i}^{(n_i)}\right)^2} \Big((1 + T_{2n}(x_i)) \ln \frac{1 - x_i}{1 + x_i} + \right. \\ & + 4 \sum_{j=1}^{n*} \frac{1}{2j-1} \left(T_{2n-2j+1}(x_i) - \cos \frac{(2j-1)(2h_i-1)}{2n} \pi \right) \Big) \\ & \left. - \frac{4\sqrt{1 - \left(x_{h_i}^{(n_i)}\right)^2}}{x_i - x_{h_i}^{(n_i)}} \sum_{j=1}^{n*} \frac{2n-2j+1}{2j-1} \cos \frac{(2j-1)(2h_i-1)}{2n} \pi \right],\end{aligned}$$

where * sign above the sum means that the last term of the sum must be divided by 2 if n is odd.

Let us $H_{\alpha_1,\dots,\alpha_p}(M_k; [-1,1]^p)$ denote the class of functions $f(x_1, \dots, x_p)$ $(-1 \le x_k \le 1, k = \overline{1,p})$,

1. satisfying the Hölder condition for each variable.

Reasoning by analogy with the work [2], we obtain the following statement.

Theorem 1. Let $f(x_1, \dots, x_p) \in H_{\alpha_1,\dots,\alpha_p}(M_r; [-1,1]^p)$, $0 < \alpha_r \le 1, r = \overline{1,p}$. Then, for the remainder term of quadrature formula (2), the estimate holds:

$$\|R_{n_1,\dots,n_p} f\|_C = O\left(\left(\sum_{r=1}^{p} n_r^{-\alpha_r} \right) \prod_{i=1}^{p} \ln^{1+\alpha_i} n_i \right).$$

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Collocation-Variational Difference Scheme for Solving the Iinverse Kinematics Problem

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The talk addresses an inverse kinematics problem. It is proposed to consider the equation that connects the rate of change of the generalized coordinates with the rate of change of the coordinates of the end effector in space as a differential-algebraic equation with a rectangular matrix in front of the principal part. A collocation-variational difference scheme is proposed to solve this problem.

Keywords: inverse kinematics problem, differential-algebraic equations, redundant manipulator, difference schemes

The talk considers the solution of an inverse kinematics problem in which it is necessary to determine the parameters of the links (rotation angles) to achieve a given position and orientation of the end effector [1]. It is proposed to consider the equation that connects the rate of change of the generalized coordinates with the rate of change of the coordinates of the end effector in space as a differential-algebraic equation with a rectangular matrix in front of the principal part. This formulation takes into account singular configurations and allows for the calculation of a stable trajectory of the end effector. A collocation-variational difference scheme is used for its numerical solution, the construction of which is based on solving a special type of mathematical programming problem. The algorithm under consideration has proven itself to solve a wide class of differential-algebraic equations, including those containing stiff components and underdetermined equations [2], [3]. Its implementation does not require pseudo-inverse of the Jacobian matrix and avoids the drawbacks of this approach, such as unstable behavior of the manipulator in singular configurations or close to them. This article applies a collocation-variational difference scheme to calculate the trajectory of a 7-axis manipulator with redundant kinematics, such as the Franka Emika Panda or its equivalent. A comparison is made with the Levenberg-Marquardt method based on Tikhonov regularization.

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On the Controllability Problem for Systems of Impulsive Functional Inclusions*

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We stady the controllability problem for systems of functional inclusions with causal operators and impulse characteristics in Banach spaces. The result of this work is a global existence theorem for systems described by functional inclusions with impulse characteristics. The proof is based on the theory of topological degree for condensing multivalued maps. As applications of the result, generalized existence theorems are obtained for systems of two important classes: first-order semilinear differential inclusions and semilinear differential inclusions of fractional order $0 < q < 1$.

Keywords: inclusion system, causal operator, functional inclusion, measure of non-compactness, fixed point, topological degree, impulse characteristics

1 The main results

We split the segment $[0,T]$ by points $0 < t_1 < ... < t_m < T$, $m \geq 1$, and introduce the notation $J = [0,T]\ \{t_1, \cdots, t_m\}$. We denote by $\mathcal{PC}([0,T];\mathcal{E})$ the space of functions $z : [0,T] \to \mathcal{E}$ such that z is continuous on J and, for each $k = 1, ..., m$, the left limit $z(t_k^-) = \lim_{\xi\to 0-} z(t_k + \xi)$ and the right limit $z(t_k^+) = \lim_{\xi\to 0+} z(t_k + \xi)$ exist, with $z(t_k^-) = z(t_k)$. The symbol $Cv(\mathcal{E})$ will denote a nonempty collection of closed convex subsets of $\mathcal{E}$.

Let $E_1, ..., E_n$ be separable Banach spaces and $E = E_1 \times \cdots \times E_n$. Define $\mathcal{C} = \mathcal{PC}\left([0,T];E_1\right) \times \mathcal{PC}\left([0,T];E_2\right) \times \cdots \times \mathcal{PC}\left([0,T];E_n\right)$.

In the present paper we consider the controllability problem for systems of functional inclusions with multivalued causal operators $\mathcal{Q}_i : \mathcal{C} \to Cv(L^{p_i}([0,T];E_i))$ and linear causal operators $\mathcal{S}_i : L^{p_i}([0,T];E_i) \to C([0,T];E_i)$, $i = 1, 2, ..., n$, of the following form:

$$\begin{cases} x_1(t) \in \mathcal{G}_1(t)x_1^0 + \Sigma_{t_k<t}\mathcal{G}_1(t-t_k)I_{k1}(x_1(t_k)) + \mathcal{S}_1 \circ \mathcal{Q}_1(x)(t) + \mathcal{S}_1 \circ \mathcal{B}_1 u(t), \quad t \in J; \\ x_2(t) \in \mathcal{G}_2(t)x_2^0 + \Sigma_{t_k<t}\mathcal{G}_2(t-t_k)I_{k2}(x_2(t_k)) + \mathcal{S}_2 \circ \mathcal{Q}_2(x)(t) + \mathcal{S}_2 \circ \mathcal{B}_2 u(t), \quad t \in J; \\ \vdots \\ x_n(t) \in \mathcal{G}_n(t)x_n^0 + \Sigma_{t_k<t}\mathcal{G}_n(t-t_k)I_{kn}(x_n(t_k)) + \mathcal{S}_n \circ \mathcal{Q}_n(x)(t) + \mathcal{S}_n \circ \mathcal{B}_n u(t), \quad t \in J; \end{cases} \tag{1}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathcal{C}$, $I_{ki} : E_i \to E_i$ are the impulse functions, control function $u = (u_1, u_2, \cdots, u_n)$ is considered in the space $L^\infty([0,T];\mathcal{U})$, $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$, $\mathcal{U}_i$ are Banach control spaces, $\mathcal{B}_i : \mathcal{U} \to E_i$ are bounded linear operators, $i = 1, \cdots, n$, and operapor-functions $\mathcal{G}_i(\cdot)$ are given by the following relation

$$\mathcal{G}_i(t) = \int_0^\infty \xi_{q_i}(\theta)U_i(t^{q_i}\theta)d\theta,\ i = 1, \cdots, n,$$

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in which

$$\xi_{q_i}(\theta) = \frac{1}{q_i}\theta^{-1-\frac{1}{q_i}}\Psi_{q_i}(\theta^{\frac{-1}{q_i}}), \quad \Psi_{q_i}(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-q_i n-1}\frac{\Gamma(nq_i+1)}{n!}\sin(n\pi q_i), \theta \in \mathbb{R}^+$$

and $U_i(\cdot)$ are C_0-semigroups.

We will consider the system under the assumption that it satisfies the initial conditions

$$x_1(0) = x_1^0 \in E_1, \quad x_2(0) = x_2^0 \in E_2, \quad \dots, \quad x_n(0) = x_n^0 \in E_n \tag{2}$$

and impulse conditions

$$x_i(t_k) = x_i(t_k^+) - I_{ki}x_i(t_k), \ k = 1, \dots, m. \tag{3}$$

By using the measure of noncompactness theory and the fixed point theory for condensing maps, we present the theorem on the existence of a mild solution to problem (1)–(3). As applications of the result, generalized existence theorems are obtained for systems of two important classes: first-order semilinear differential inclusions and semilinear differential inclusions of fractional order $0 < q < 1$.

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Feedback Minimum Principle and Approximate Maximum Principle for Discrete Optimal Control Problems

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We consider two approaches to necessary optimality conditions in discrete dynamic problems: the Feedback Minimum Principle and the Approximate Maximum Principle. The aim is to combine them into a hybrid method that should be applicable to discrete problems without any additional assumptions about dynamical systems.

Keywords: necessary optimality conditions, maximum principle, discrete dynamical systems

This talk concerns the following discrete optimal control problem (P):

$$x_{t+1} = f_t(x_t, u_t), \quad u_t \in U_t, \quad t = 0, \dots, T-1, \quad x_0 \quad \text{is fixed,}$$
$$J[u] = l(x_T) \to \inf,$$

where functions f_t and l are smooth with respect to the first argument, sets U_t are compact. The state vectors $x_t \in R^n$, and control vectors $u_t \in R^m$.

It is widely known that the Discrete Maximum Principle — the direct analogue of classical Pontryagin Maximum Principle — is false for nonlinear problems, which are not satisfy for special convex assumptions [1,2,3]. In general case, this result doesn't give even necessary optimality conditions.

The research is devoted to develop a method, that uses Feedback Minimum Principle [4] and Approximate Maximum Principle [5] to obtain a necessary optimality condition, which is valid for problem (P) without any convex demands.

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Terminal Constraints in the Feedback Minimum Principle: Challenges and Numerical Implementation

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This report examines the key features of incorporating terminal constraints into the feedback minimum principle (FPM), a nonlocal necessary optimality condition for optimal control problems. Unlike problems with a free right endpoint, the presence of terminal conditions introduces significant challenges due to the discontinuous nature of feedback descent controls and the need for singular limit transitions. We discuss several methods to overcome these difficulties in one class of optimal control problems. Based on these methods, iterative numerical algorithms for optimal impulsive processes are proposed.

Keywords: feedback minimum principle, terminal constraints, optimal control, nonlocal optimality conditions, feedback descent controls

The feedback minimum principle (FPM) for optimal control problems was originally established for problems with a free right endpoint of the trajectory. It represents a nonlocal necessary condition for global optimality that strengthens the classical Pontryagin maximum principle [1,2,3,4,5]. A generalization of the FPM to optimal control problems with terminal equality constraints was considered in [6,7].

Direct extension of the FPM to problems with terminal constraints encounters serious difficulties. The feedback descent controls used in the FPM are discontinuous, which leads to the need for singular limit transitions when analyzing the behavior of trajectories at the right endpoint. This makes the practical application of the method for analytical study of model examples extremely problematic. To overcome these difficulties, V.A. Dykhta [7] combines two ideas: the method of support majorants and the modified Lagrange function with a quadratic penalty. Instead of the original constrained problem, a family of unconstrained problems is considered, where the terminal conditions are incorporated into the objective functional as a penalty term. The modified Lagrangian has the property of being a support majorant: it coincides with the original functional at the point under study and does not exceed it everywhere else.

In this work, we discuss other techniques for accounting for terminal constraints in optimality conditions of the FPM type. The main focus is on two methods proposed in [8,9,10], applied to optimal impulsive control problems, in which terminal constraints arise naturally in the space-time representation of impulsive processes. In the optimal impulsive control problem, after a transformation to the reduced problem by means of a discontinuous time reparameterization, a terminal constraint on the auxiliary state variable y appears: $y(T) = y_T$. This constraint is crucial because it ensures the equivalence between the original impulsive problem and the reduced problem with ordinary controls. A direct application of the feedback minimum principle (in its classical form designed for free-endpoint problems) to the reduced problem is impossible, since an arbitrary closed-loop system with an extremal feedback

$$v(t, x) \in V_\xi(x, \psi(t)),$$

generally speaking, does not guarantee the satisfaction of the terminal condition $y(T) = y_T$. In other words, trajectories generated by extremal feedback controls may not reach the prescribed value y_T at the final time.

We propose two modifications of the extremal multivalued mapping that take into account the current discrepancy in the variable y. These modifications allow us to enforce the terminal constraint while preserving the improving property of the feedback control.

Based on these corrected extremal mappings, a counter-positive version of the nonsmooth feedback maximum principle yields a conceptual iterative algorithm for optimal control, which can be used for numerical implementation of the optimal impulsive control problem. Since numerical analysis of the transformed model requires its discretization, we consider a discrete-time counterpart of the transformed problem. For this class of optimization problems, a discrete-time version of a nonlocal necessary optimality condition was derived in [9,10]. Based on this optimality condition, an iterative numerical algorithm is developed.

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A Comparative Study of Swarm Intelligence Algorithms Applied to the Morse Potential Optimization Problem

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This paper addresses the global minimization of the Morse potential, a canonical benchmark for evaluating optimization methods on atomic and molecular cluster problems. Over twenty swarm intelligence algorithms are compared both in standalone mode and in combination with the L-BFGS local search procedure on clusters of up to 80 atoms (240 variables). The results demonstrate the practical efficiency of the considered metaheuristic approaches and highlight the benefit of hybridizing population-based global search with local refinement.

Keywords: global optimization, Morse potential, swarm intelligence, metaheuristics, atomic clusters, L-BFGS

Growing interest in the synthesis and characterization of ultrafine atomic and molecular structures has stimulated intensive research into efficient methods for determining cluster geometries that correspond to minimal potential energy. Problems of this class are typically formulated as the minimization of highly multimodal, nonconvex objective functions whose number of local optima grows extremely rapidly with the dimension of the search space. Locating a global minimum of such functions remains one of the most demanding challenges in theoretical and applied optimization. The present work focuses on the minimization of a multimodal function defined by the Morse potential [1].

The past decade has seen sustained growth in the development and analysis of numerical methods oriented toward global optimization of complex landscapes. A broad spectrum of population-based algorithms has been proposed in the literature, each exhibiting distinctive computational properties and targeted application areas. A prominent subfamily of such approaches comprises metaheuristic schemes grounded in behavioral or biological analogies found in nature, particularly those falling under the umbrella of swarm intelligence [2,3]. The widening adoption of these methods is attributable to the continuing growth of available computational resources and the pressing demand for scalable solvers capable of handling high-dimensional real-world problems.

The paper reports a systematic numerical investigation of more than twenty contemporary swarm-based global optimization algorithms applied to the Morse potential minimization problem. The collection of methods under study includes, among others, the Firefly Algorithm, the Coyote Optimization Algorithm, the Gorilla Troops Optimizer, the Tuna Swarm Algorithm, and the Ant Lion Optimizer [2,3]. Benchmark tests were conducted on cluster configurations containing as many as 80 atoms, corresponding to an optimization problem with 240 continuous variables. For each algorithm, two experimental scenarios were evaluated: independent operation of the population-based search and a hybrid scheme in which the swarm method was coupled with the L-BFGS gradient-based local refinement procedure. Numerical results illustrating the comparative performance of all considered approaches are presented and discussed.

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Finding Eigenvalues on Physics-informed Radial Basis Function Networks*

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A method for finding eigenvalues using physics-informed radial-based function networks is proposed. Classical numerical methods—finite-difference, finite-element, and spectral—face a number of limitations: the need to construct a grid in a domain of complex shape, the exponential growth of computational complexity with the spatial dimension, and the difficulty of solving nonlinear eigenvalue problems. Unlike traditional methods that require grid construction, PINNs provide a meshless approximation of eigenfunctions [1]. The solution is represented as a continuous differentiable function parameterized by the weights of the neural network. This allows one to calculate eigenfunctions at any point in the domain without interpolation and naturally account for complex geometries without mesh generation. Classical methods often require an excessively fine grid to resolve high-frequency eigenvalues. PINNs demonstrate the ability to adapt to multi-scale features of the solution. In [2], using neural tangential kernel (NTK) theory, it was shown that PINNs are capable of learning various frequency components of the solution, and spectral bias can be controlled through architectural modifications. PINNs allow eigenvalue problems to be formulated as variational problems, where the first few eigenfunctions can be found simultaneously using orthogonalizing terms in the loss function. However, solutions using classical PINNs can be time-consuming. The use of physics-informed radial basis function networks [3] can increase the convergence rate and improve the solution accuracy. In this work, not

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only the weights of the physics-informed radial basis function networks are tuned, but also nonlinear parameters: the centers and widths of the radial basis functions.

Keywords: eigenvalues, partial differential equations, physics-informed radial basis function networks

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On the Selection of a Prior Distribution for Bayesian Estimation of a Measurand Near Natural Limits

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The paper proposes a method for the Bayesian estimation of a measurand located near natural limits. To address the lack of precise a priori information, a weighted least squares approach is developed to construct a prior distribution based on limited certificate (specification) data. The proposed technique ensures metrological consistency by keeping the results within the allowed domain and is validated through numerical experiments.

Keywords: bayesian estimation, prior distribution, natural limits, measurand, weighted least squares

When measuring quantities with natural limits, the classical approach based on a normal error model can yield estimates that fall outside the permissible range. This issue is particularly acute near a boundary, where the standard uncertainty is comparable to the distance from it [1]; in such cases, the coverage interval may extend beyond the domain of the measurand. The Bayesian approach inherently keeps the result within the allowed domain, as the prior distribution is defined with a support that coincides with the natural limits [2]. However, in practice, the selection of the prior distribution remains a key challenge: often, only indirect and imprecise specification data (e.g., from reference material certificates or prior experience with a manufacturer) are available.

The paper adopts a standard metrological measurement model, where the measurement result is related to the true value by an additive normal error with zero mean and a known standard uncertainty. The true value is known to belong to a given interval, which, without loss of generality, can be mapped to the unit interval $[0, 1]$ via a linear transformation. Diverse statements can be extracted from specification data: the most probable (nominal) value, as well as one-sided or two-sided coverage intervals for a given probability level (possibly for several). These data are not perfectly accurate; they may contain errors or contradict one another. Therefore, a method is required to construct a prior distribution that best satisfies all conditions without requiring their strict fulfillment.

The prior density must be defined on a fixed (typically finite) interval. Natural candidates include the families of beta distributions, two-sided power (TSP) distributions, or truncated normal distributions. The choice of a specific family can be determined by its flexibility and computational convenience, as well as specific physical considerations. To estimate the parameters of the chosen family, a weighted least squares (WLS) method is proposed. Each expert condition (mode, quantile, etc.) is associated with a residual between the theoretical value for the given distribution and its estimate. The problem is reduced to minimizing the weighted sum of squared residuals; this method is robust to data inconsistencies and provides a compromise solution without requiring strict compliance with all conditions.

Additionally, entropy regularization can be incorporated into the objective function: a penalty proportional to the negative differential entropy of the distribution. This encourages the selection of a higher-entropy distribution, making the parameter estimation more objective, especially when the number of conditions is small or they are weakly informative. The regularization coefficient serves as a tool to stabilize the solution in cases of deficient or highly contradictory prior data. The numerical implementation involves parameter optimization using gradient-based methods subject to constraints (e.g., the parameters of a beta distribution must be positive). For distributions without an analytical form for entropy, numerical integration can be applied. Once the prior distribution is constructed, the posterior distribution for a given measurement is computed as the product of the prior density and the normal likelihood, followed by normalization. This allows for the calculation of the posterior density and its statistical characteristics (mean, median, mode) — which always lie within the natural limits — as well as coverage intervals for a given probability.

The efficiency of the proposed method was validated through numerical experiments. The approach combines metrological consistency (accounting for limits), flexibility (arbitrary parametric families), and robustness to imprecise certificate data (WLS). It can be effectively used for the correct processing of measurement results near the natural limits of a measurand.

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New Approach to Epidemic Nonconvex Optimal Control Problem*

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We address an Optimal Control (OC) Problem with a state-nonlinear control system (CS) and a terminal cost functional given by a convex function, so that the nonconvexity of the OC Problem is produced only by a specific state-nonlinear CS. First, the OC Problem is reduced to a variational problem without constraints with the help of the Exact Penalization Theory and usual penalty function. Further, we construct an estimation of the classical penalty function with the help of a special form of CS, given by state-nonlinear system of ODE's with the state-DC functions. Using the DC-decomposition of the DC-data, we proposed the Global Optimality Conditions (GOCs) of necessary and sufficient form for this kind of nonconvexity. The GOCs possesses the "constructive property": if the "principle inequality" of GOCs is violated, then there is a possibility to improve the control under scrutiny and to get a better value of the cost functional. It allows not only to ameliorate stationary controls (i.e. satisfying the PMP), but to construct numerical method for nonconvex OC Problem.

Keywords: Nonconvex Optimal Control Problem, Exact Penalization Theory, DC-decomposition, Global Optimality Conditions.

On Epidemic Nonconvex Optimal Control Problems*

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The paper considers an epidemic optimal control problem governed by a state-linear control system of ordinary differential equations and a Bolza-type cost functional possessing a DC decomposition. The nonconvexity of the problem is accumulated in the state component of the functional, which leads to the existence of multiple critical processes and limits the applicability of Pontryaguin's principle for global optimality. To overcome this difficulty, a local search scheme based on partial linearization of the nonconvex part of the functional with respect to the state is proposed. At each iteration, an auxiliary optimal control problem with a state-convex functional is constructed and solved, while the control system remains unchanged. The method generates a sequence of admissible controls by solving a

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series of partially linearized problems and uses a stopping criterion based on the decrement of the linearized functional and the residue of Pontryagin's principle. The convergence analysis establishes almost monotonic decrease of the objective functional values and convergence of the sequence to a finite limit, as well as convergence of the corresponding state trajectories under additional convexity assumptions. A computational experiment on epidemic models of increasing complexity demonstrates that the proposed scheme efficiently computes critical processes within a small number of iterations and computational time, confirming its applicability to epidemic optimal control problems..

Keywords: epidemic OC problems, state-linear control system, the cost Bolza DC functional, Pontryaguin's principal, local search scheme

1 The main results

We consider an epidemic optimal control problem governed by a linear control system of ordinary differential equations [2,4,6]

$$\dot{x}(t) = A(t)x(t) + B(u(t),t) \quad \overset{\circ}{\forall} t \in T, \quad x(t_0) = x_0. \tag{1}$$

$$u(\cdot) \in \mathcal{U} := \left\{ u(\cdot) \in L^r_\infty(T) \mid u(t) \in U \quad \overset{\circ}{\forall} t \in T =]t_0, t_1[\right\}. \tag{2}$$

In epidemic applications [1], the state vector $x(t) \in \mathbb{R}^n$ describes the distribution of the population over epidemic groups, the matrix $A(t)$ reflects natural epidemic transitions such as infection, recovery, or death, and the vector-function $B(u(t),t)$ represents anthropologic impacts, for instance treatment or vaccination. Since the components of the state vector represent percentages of the total population, the conservation condition takes place

$$\sum_{i=0}^{n} x_i(t) = 100 \quad \forall t \in T. \tag{3}$$

The optimal control problem is defined by the Bolza-type functional

$$(\mathcal{P}): \quad J(u) := \varphi_1(x(t_1,u)) + \int_T \varphi(x(t,u),u(t),t)dt \downarrow \min_u, \quad u(\cdot) \in \mathcal{U}, \tag{4}$$

where

$$\left.\begin{aligned} \varphi_1(x) &= g_1(x) - h_1(x) \\ \varphi(x,u,t) &= g(x,t) - h(x,t) + f(u,t); \end{aligned}\right\}. \tag{5}$$

Here $g_1(\cdot)$, $h_1(\cdot)$, $g(\cdot,t)$, and $h(\cdot,t)$ are convex with respect to the state variable. Therefore the terminal and running terms are state DC-functions, and the objective functional admits the DC decomposition

$$J(u) = G(x(\cdot),u(\cdot)) - F(x(\cdot)), \tag{6}$$

where

$$\left.\begin{aligned} &(a): \quad G(x(\cdot),u(\cdot)) := g_1(x(t_1)) + \int_T [g(x(t),t) + f(u(t),t)]\,dt, \\ &(b): \quad F(x) := h(x(\cdot)) + \int_T h(x(t),t)dt; \end{aligned}\right\} \tag{7}$$

Thus, the nonconvexity of the original problem is accumulated by the functional $F(\cdot)$. This is important because in such problems Pontryagin's principle is no longer sufficient for global optimality, and the problem may possess a large number of locally optimal and stationary processes. [2]

For the functional $F(\cdot)$, the generalized differential is given by next way

$$\langle\langle \nabla F(y(\cdot)), x(\cdot)\rangle\rangle := \langle \nabla h_1(y(t_1)), x(t_1)\rangle + \int_T \langle \nabla h(y(t),t), x(t)\rangle dt, \tag{8}$$

besides, the corresponding convexity inequality has the form

$$\langle\langle \nabla F(y(\cdot)), x(\cdot) - y(\cdot)\rangle\rangle \leq F(x(\cdot)) - F(y(\cdot)), \tag{9}$$

which allows us to consider the next partially linearized optimal control problem

$$\left.\begin{aligned} \Phi_y(x(\cdot),u(\cdot)) := \Phi_y(u) := G(x(\cdot),u(\cdot)) - \langle\langle \nabla F(y(\cdot)), x(\cdot)\rangle\rangle = \\ = g_1(x(t_1)) - \langle \nabla h_1(y(t_1)), x(t_1)\rangle + \int_T [g(x(t),t) + \\ + f(u(t),t) - \langle \nabla h(y(t),t), x(t)\rangle +]\, dt \downarrow \min_{u(\cdot)},\ u(\cdot) \in \mathcal{U}; \end{aligned}\right\} \quad (\mathcal{P}L(y)) \tag{10}$$

along the same control system (1)–(2). Since the functional Φ_y is state convex, the auxiliary problem is more convenient for numerical solution, in particular, with the help of methods based on Pontryaguin's principle. The linearization is performed only with respect to the nonconvex part of the cost functional, while the state-linear control system remains unchanged. [3] [5]

In this case, a local search scheme is developed for solving the considered epidemic optimal control problem. Namely starting from an initial admissible control $u^0(\cdot) \in U$, a sequence of controls $\{u^s(\cdot)\} \subset U$ is constructed, when at each iteration s, the next control $u^{s+1}(\cdot)$ is obtained as an approximate solution of the partially linearized problem (PL_s), linearized at the current state trajectory $x^s(\cdot) = x(\cdot, u^s)$. Thus, the method is based on the consecutive solution of auxiliary optimal control problems obtained by linearization of the state nonconvex part of the cost functional with respect to the state.

The stopping criterion of the algorithm is formulated in terms of the decrement of the linearized functional Φ_s and the residue of Pontryagin's principle. The proposed scheme ensures a monotone behavior of the objective functional up to a summable error and allows one to construct a sequence of feasible processes that approximate critical processes of the original problem.

The convergence analysis shows that the sequence of objective functional values $\{J(u^s)\}$ is almost monotonically nonincreasing and converges to a finite limit. Moreover, under additional assumptions of strong convexity of the corresponding functions in the DC decomposition, convergence of the state trajectories is established in the sense of vanishing successive differences.

A computational experiment was carried out on epidemic optimal control problems with increasing complexity, including the ISR, $ISRD$, and I_1I_2SRD models. The numerical results demonstrate that the developed local search scheme, combined with an inner method based on Pontryagin's principle, efficiently computes critical controls under control constraints and a nonconvex terminal cost. In most cases, the method converges within a small number of iterations and requires only a few seconds of computational time.

These results demonstrate that the developed approach is promising for practical application and can be used as a basis for the development of global search schemes for epidemic optimal control problems.

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On One Variant of a Difference Scheme with Weights for a Two-dimensional Differential-Algebraic System*

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This paper examines a linear, two-dimensional, first-order differential-algebraic system of mixed type with given initial-boundary conditions. For its numerical solution, a difference scheme based on a combination of a spline-collocation difference scheme and a weighted scheme is proposed.

Keywords: differential-algebraic system, spline, difference scheme with weights

This paper examines a first-order, linear, two-dimensional, mixed-type differential-algebraic system with given initial-boundary conditions. To solve it numerically, a difference scheme is constructed based on a combination of a spline-collocation difference scheme and a difference scheme with weights. Using the spline-collocation method, the initial differential-algebraic system is associated with a differential-operator equation, which is an algebraic-differential system. It is assumed that this system has an index no greater than one. A weighted scheme is used for its numerical solution. The resulting difference scheme is analyzed, and stability conditions for its initial-boundary conditions and right-hand side are obtained.

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Optimal Arc Shape in a Potential Flat Flow

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This work illustrates an application of the geometrically functional method for solving plane optimization problems with a free boundary. The essence of the method lies in the ability to relate the solution of the original problem in the velocity field of the potential flat flow to the solution of Dirichlet problem for a special harmonic function [1,2]. As an example, the shape is chosen to be a part of a circular arc, convex towards the oncoming flow. An arbitrary angle of the circular arc is a control parameter in solving various extremal problems. In the first such problem, the functional is the time of washing over the specified arc-shaped object. In the second, it is the pressure force on the same shape.

Keywords: Euler's equations, incompressible fluid, shape optimization

1 The method and results

In the plane $\mathbb{R}_x \times \mathbb{R}_y$ we describe the shape optimization problem of the object in a continuous potential flat liquid flow. At the time t the velocity $V(t) = dz/dt$ of the particle with coordinates $(x(t), y(t)) \simeq z(t) = x(t) + iy(t) \in \mathbb{C}$ equals $\nabla\varphi(x(t), y(t))$. Let us consider the case when the flow velocity at the infinite point is equal $\mathbf{V}_\infty = V_\infty(1, 0)$, and the object shape OM is an arc (fig.1). It is interesting to find the angle θ and provide the minimal moving time $T > 0$ of the particle along the OM with geometric condition $|OM| = R\theta$, where R is the arc circle radius. Another our goal in exploring the extremal pressure force $\mathbf{F} = (F_x, F_y)$.

The current domain (with $y > 0$) is denoted by Ω, and let it be considered as a subdomain of a complex set $\mathbb{C}$. Inside this simply connected domain, we introduce an analytic function $w = \varphi + i\,\psi$, with zero value at the O point. Lines level of the function φ and ψ is called equipotential and stream line, respectively. In the stream line $\psi = 0$ let us use the natural parameter ξ, computed from the point O and increased toward the growth of φ. A point (fluid particle) in the stream line $\psi = 0$, concerned with ξ, is denoted $P(\xi)$ and has coordinates z_ξ. ***Our goal*** is to find the angle $\theta \in]0, \pi/2[$ to provide the extremal time and force, concerned with the arc from O to M. Note that $\varphi_O = \varphi_{P(0)} = \varphi(0, 0) = 0\,,\ \varphi_M = a > 0\,.$

Obviously, that $w : z \mapsto w(z) = \varphi(x, y) + i\ \psi(x, y)$ is a single-sheeted mapping from Ω to $\mathbb{C}_+ = \{\mathrm{Im}\ Z > 0\}$. So there are analytic functions $Z : w \mapsto z$, such as $Z(w(t)) = z(t)$, and a relevant one $\ln dZ/dw = A(\varphi, \psi) + i\ B(\varphi, \psi)$, where $A = \ln|dZ/dw|$, and B is a continuous branch of $\mathrm{Arg}(dZ/dw)$, provided by a single-valued function because $\mathbb{C}_+$ is a simply connected domain.

According to the geometric sense of the $\mathrm{Arg}(dZ/dw)$, any point on the stream line $\psi = 0$ has $B(w)|_{\psi=0} = B(\varphi) = \theta * (1 - \varphi/a) \cdot \chi[0, a](\varphi)$ with the function B harmonic inside the domain $\mathbb{C}_+$. Since B is defined in a half-plane domain $\psi > 0$ and has such boundary conditions at $\psi = 0$, it is determined by the Poisson integral formula. The harmonic A conjugate to B can be find from the Cauchy-Riemann conditions, and then $Z'(w) = e^A e^{iB}$ is obtained. The next observation provide the way to find $V = \dot{z}$

$$\dot{z}(t) = \varphi_x + i\,\varphi_y = \overline{\varphi_x - i\,\varphi_y} = \overline{\varphi_x + i\,\psi_x} = \overline{\frac{dw}{dZ}} = \overline{\left(\frac{dZ}{dw}\right)^{-1}} = \overline{Z'(w)^{-1}} = e^{-A+iB}$$

Finally, the *const* to determine φ and ψ has been found from the condition at the infinity point of flow. Eventually,

$$v\big|_{\psi=0} = \dot{z}\big|_{\psi=0} = e^{\theta/\pi}\left|\frac{\varphi/a}{1-\varphi/a}\right|^{\theta*(1-\varphi/a)/\pi} e^{i\,\theta*(1-\varphi/a)}. \tag{1}$$

The function φ meets the differential equation derived from the fact $Z(w(t)) = z(t)$. Whence $Z'(w)\dot{w} = \dot{z}$ or $e^{A+iB}\dot{w} = e^{-A+iB}$ and, consequently with $\psi = 0$, $\dot{\varphi} = e^{-2A}$. The last equation with the $\varphi(0) = 0$ gives us the Cauchy problem and opportunity to find an expression for the time $T(a, \theta)$. And the geometrical restriction $|OM| = R\theta$ now can be expressed by the formula $\Phi(a, \theta) = 0$ that gives an implicitly defined function $a = a(\theta)$. Further exploration is possible with $T(\theta, a(\theta))$ by using numerical methods.

As for the pressure force, it can be found from the integration of the pressure at each point in the stream line $\psi = 0$ on the arc. The pressure is represented by the Bernoulli integral with the denotions $p = (0.5 + p_\infty/(\rho V_\infty^2)) - 0.5v^2 = \mathbf{c} - 0.5v^2$.

Desired values and extremal points are illustrated in the right part of Fig.1. It should be noted that F_x component of the force is a negative constant due to the Dalembert paradox.

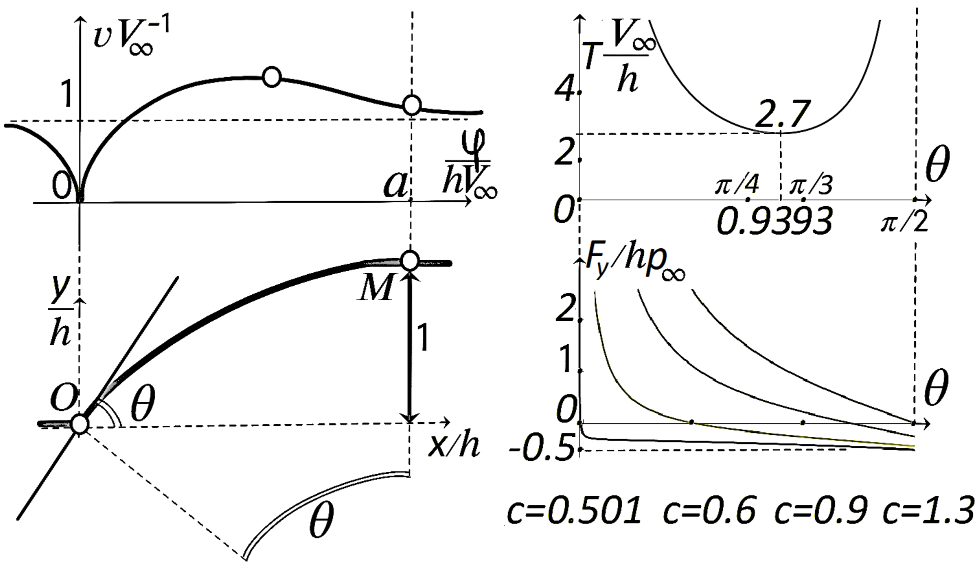

Figure 1: The shape in the flow and result for the boundary velocity field above it. At right the Time and the Force.

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Optimality Conditions and Optimization Algorithms in Infinite-Dimensional Spaces

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The problem of minimizing functionals in infinite-dimensional spaces is considered. It is shown that traditional methods can only ensure integral convergence of controls $u(\tau), \tau \in S$ to the optimum $u_*(\tau)$ in $L_2(S)$. New necessary conditions for optimality in the vicinity of the optimum [1] are formulated for infinite-dimensional space. Based on these conditions, extreme algorithms with adjustable descent direction are obtained [2,3,4]. They can ensure fast uniform convergence of controls to the exact solution on a space-time set S, or on $S_\Delta \subset S$, where pointwise convergence is possible in principle.

Keywords: necessary conditions for optimality, extreme algorithms, gradient, infinite-dimensional optimization

1 The main results

Theorem(*Strong* NCO) If a strictly convex quadratic functional $J(u)$, $u \in L_2(S)$ has a minimum at the point u_*, and the sequence of controls u^k is such that $\left|u_* - u^k\right| \xrightarrow{k\to\infty} 0$ uniformly on S_Δ, then necessary:

$$\left|\nabla J(u^k;\tau)\right| \xrightarrow{k\to\infty} 0 \text{ uniformly on } S_\Delta. \tag{1}$$

Remember that the traditional *weak* NCO is $\|\nabla J(u^k;\tau)\|_{L_2(S)} \xrightarrow{k\to\infty} 0$. It ignores the direction (variety of functions $\nabla J(u^k;\tau)$ with the same norm) of moving to u_*. A strong NCO for a quadratic functional sets the direction from u^0 directly to a small vicinity of u_*.

From (1) follows the adjustable descent direction method based on the gradient (ADDMg):

$$u^{k+1}(\tau) = u^k(\tau) - b^k\alpha(\tau)\nabla J(u^k;\tau) \text{ uniformly on } S_\Delta, \quad k = 0, 1..., \tag{2}$$

where $\alpha(\tau) \in C^1_+(S_\Delta)$ is a parameter that controls the direction of descent to ensure the uniform convergence required for strong NCO. For example, in the first step, you need to:

$$u^0(\tau) \xrightarrow{\text{uniform step}} u^1(\tau), \quad \tau \in S_\Delta,$$
$$\nabla J(u^0;\tau) \xrightarrow{\text{uniform change}} \nabla J(u^1;\tau), \quad \tau \in S_\Delta.$$

Under these conditions, for a convex functional and a suitable $u^1(\tau)$, we can define:

$$\alpha(\tau) = \left|\frac{u^1(\tau) - u^0(\tau)}{\nabla J(u^0;\tau)}\right|, \quad \text{sgn} J(u^0;\tau) = const, \quad \tau \in S_\Delta. \tag{3}$$

2 Test demonstrations

Consider a test example for the synthesis of dynamic optimal control $u_*(t)$ in a one-dimensional heat transfer problem. In such problems, the optimal system state, the temperature T_*, is known, and it is necessary to find the corresponding control $u_*(t)$.

We need to stabilize the temperature T_* at the left boundary x_a for the heat flow $u(t)$ at the right boundary x_b:

$$J(u) = \int_{t_0}^{t_1} \left(T(x_a, t) - T_*(t)\right)^2 dt \to \min . \tag{4}$$

A test u_* was set, and T_* was found. Optimization started with $u^0 = 400$ and ended when $||u^k - u^{k-1}||/||u^{k-1}|| < 10^{-6}$.

Fig. 1a shows the optimization results by traditional methods when $\alpha(t) = 1$: Steepest descent method (SDM) and Limited memory BFGS (L-BFGS) as a finite-dimensional method at the nodes of the computational grid in S. Fig. 1b shows the optimization results by the method (2) when $\alpha(t) = 0.2u^0/\left|\nabla J(u^0;t)\right|$.

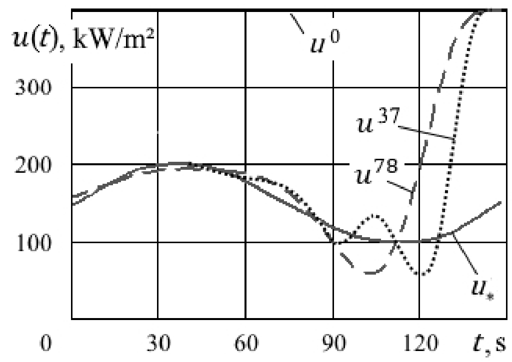

Figure 1: SDM (dotted line) and L-BFGS (points)

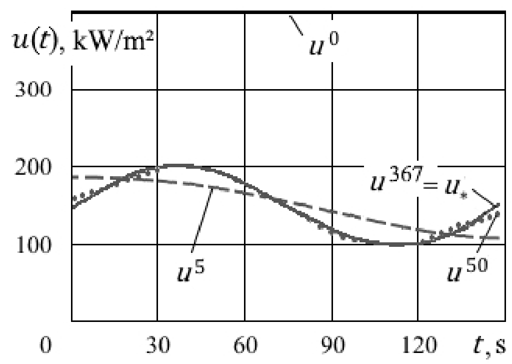

Figure 2: ADDMg

Figure 3: Heat flow optimization results

Uniform convergence for the ADDMg (2) was obtained. The functional (4) at $k = 50$ decreased by 2 orders more than the SDM and at $k = 367$ by 5 orders more. A large number of optimization examples for applied infinite-dimensional problems can be found in [2].

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Optimal Control of an Age-Structured Population with Time-Delay Effects

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This paper considers an optimal control problem for an age-structured population model with time-delay effects. The population dynamics are described by a partial differential equation with respect to time and age, incorporating age-dependent birth and death rates. The control objective is to determine a time-dependent control function that minimizes a given performance functional, which measures the deviation of the population distribution from a desired target state at the final time. The admissible control is assumed to be continuously differentiable and subject to given constraints. The proposed formulation allows taking into account delay effects in the population dynamics and can be applied to various problems in population management and biological systems.

Keywords: age-structured population model, optimal control problem, population dynamics

1 The main results

This work addresses the problem of controlling a population while taking into account time-delay effects. The population dynamics equation has the following form [1]:

$$\begin{cases} \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \tau} = -d(\tau, t)\, x, \\ \\ x(0,t) = \int_0^\infty b(\tau,t)\, x\, d\tau, \\ \\ x(\tau, 0) = \varphi(\tau). \end{cases} \qquad (1)$$

Here, t is the initial time at which the population dynamic is observed over a time interval $T = [0, t_k]$, τ is the age of individuals, $\tau \in [0, \tau_k]$, τ_k is the maximum lifespan, $x(\tau, t)$ is the age-density function of the population, and $b(\tau, t)$, $d(\tau, t)$ are the birth and death rates, respectively.

The optimal control problem is reduced to minimizing the functional [2]:

$$J(u) = \int_0^{\tau_k} \varphi(x(\tau, t_k), \tau)\, d\tau, \tag{2}$$

where $\varphi(x, \tau) = \frac{1}{2}\big(x(\tau, t_k) - \bar{x}(\tau)\big)^2$ Here, $\bar{x}(\tau)$ is a given function.

The optimal control $u(t)$ is continuously differentiable on the interval T and satisfies the following constraints:

$$u(t) \in U, \quad t \in T. \tag{3}$$

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Differential Invariants of the WDVV Equation

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Lie algebra of point symmetries of the WDVV equation is studied. The corresponding field of differential invariants of the WDVV equation is found.

Keywords: WDVV equation, symmetries, differential invariants

1 The main results

We study differential invariants of the WDVV equation (also know as the associativity equation), which arises in classification of three-dimensional Frobenius manifolds (see, [1]). This is a PDE on function $u(x, y)$ of the form

$$u_{yyy} + u_{xxx}u_{xyy} - u_{xxy}^2 = 0.$$

Theorem 1. *Vector fields*

$$\begin{aligned}
&X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_u,\\
&X_4 = x\partial_u, \quad X_5 = y\partial_u, \quad X_6 = xy\partial_u,\\
&X_7 = y\partial_y - u\partial_u, \quad X_8 = x\partial_x + 4u\partial_u, \quad X_9 = x^2\partial_u,\\
&X_{10} = y^2\partial_u, \quad X_{11} = 2y\partial_x + x^3\partial_u,\\
&X_{12} = xy\partial_x + y^2\partial_y + \left(2uy + \frac{x^4}{8}\right)\partial_u
\end{aligned}$$

generate a Lie algebra $\mathfrak{g}$ of point infinitesimal symmetries of the WDVV equation.

We prove that the corresponding group of symmetries acts algebraically on the solutions of the WDVV equation.

Thus, according to the Lie–Tresse theorem [2] the following invariants generate the field of differential invariants of the WDVV equation.

$$\mathcal{J}_1 = \frac{\mathcal{R}_5^2}{\mathcal{R}_2^3} = \frac{\left(2u_{3,0}{}^3 + 18u_{2,1}u_{3,0} + 27u_{1,2}\right)^2}{\left(u_{3,0}{}^2 + 6\,u_{2,1}\right)^3},$$
$$\mathcal{J}_2 = \frac{\mathcal{R}_6^2}{\mathcal{R}_3^2\mathcal{R}_2} = \frac{\left(2u_{3,0}{}^2u_{4,0} + 6u_{2,1}u_{4,0} + 6u_{3,0}u_{3,1} + 9u_{2,2}\right)^2}{\left(u_{3,0}{}^2 + 6u_{2,1}\right)\left(9u_{3,0}u_{4,0} + 27u_{3,1}\right)^2},$$
$$\mathcal{J}_3 = \frac{\mathcal{R}_1^2\mathcal{R}_2^3}{\mathcal{R}_3^4} = \frac{u_{5,0}^2\left(u_{3,0}{}^2 + 6u_{2,1}\right)^3}{\left(u_{3,0}u_{4,0} + 3\,u_{3,1}\right)^4},$$
$$\mathcal{J}_4 = \frac{\mathcal{R}_4\mathcal{R}_2}{\mathcal{R}_3^2} = \frac{\left(u_{3,0}u_{5,0} + u_{4,0}{}^2 + 3u_{4,1}\right)\left(u_{3,0}{}^2 + 6u_{2,1}\right)}{\left(u_{3,0}u_{4,0} + 3\,u_{3,1}\right)^2},$$
$$\mathcal{J}_5 = \frac{\mathcal{R}_2\mathcal{R}_7^2}{\mathcal{R}_3^4} = \frac{\left(u_{3,0}^2u_{5,0} + \left(4u_{4,0}{}^2 + 6u_{4,1}\right)u_{3,0} + 12u_{3,1}u_{4,0} + 9u_{3,2}\right)^2}{\left(u_{3,0}u_{4,0} + 3u_{3,1}\right)^4}\left(u_{3,0}^2 + 6u_{2,1}\right),$$

where the following functions are the relative invariants.

$$\begin{aligned}
&\mathcal{R}_1 = u_{5,0}, \quad \mathcal{R}_2 = u_{3,0}^2 + 6u_{2,1}, \quad \mathcal{R}_3 = u_{3,0}u_{4,0} + 3u_{3,1},\\
&\mathcal{R}_4 = u_{3,0}u_{5,0} + u_{4,0}^2 + 3u_{4,1}, \quad \mathcal{R}_5 = 2u_{3,0}^3 + 18u_{2,1}u_{3,0} + 27u_{1,2},\\
&\mathcal{R}_6 = u_{3,0}^2u_{5,0} + \left(4\,u_{4,0}{}^2 + 6\,u_{4,1}\right)u_{3,0} + 12\,u_{3,1}u_{4,0} + 9\,u_{3,2},\\
&\mathcal{R}_7 = 2u_{3,0}{}^2u_{4,0} + 6u_{2,1}u_{4,0} + 6u_{3,0}u_{3,1} + 9\,u_{2,2}.
\end{aligned}$$

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Multi-method Optimization of Control

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For solving a complex optimal control problem, we first apply reduction to a finite-dimensional problem and then construct a multi-method algorithm for finding its solution.

Keywords: optimal control, multi-method optimization, reduced gradient, linear programming, Lagrange function

1 Problem statement

Let a controlled process be given, depending on parameters:

$$\dot{x} = f(x, u, w, t),\ x(t) \in E^n,\ u(t) \in E^r,\ t \in T = [t_0, t_1],\ x(t_0) = \Theta(v),\ w \in R^p,\ v \in R^n \quad (1)$$

with terminal conditions

$$I_i(u) = h_i(x(t_1)) = 0, \quad i = \overline{1, m}, \tag{2}$$

and phase constraints

$$J_i(u, v) = g_i(x(t), t) = 0, \quad t \in T, \quad i = \overline{1, s}. \tag{3}$$

Control and parameters are subject to the following constraints:

$$c_i(u, t) = 0, \quad t \in T, \quad i = \overline{1, l}, \tag{4}$$

$$u^L(t) \le u(t) \le u^U(t), \quad t \in T, \tag{5}$$

$$v^L \le v \le v^U, \quad w^L \le w \le w^U. \tag{6}$$

Among controls and parameters satisfying constraints (4)–(6), it is required to find those that ensure fulfillment of conditions (3) for the controlled process (1) and bring it to a point in the phase space where conditions (2) are satisfied with given accuracy, and the functional

$$I_0(u) = \phi(x(t_1)) \tag{7}$$

attains its minimum value.

2 Algorithm of the reduced gradient method

Introducing vector notation for equalities (2)–(4), we construct the modified Lagrange function for problem (1)–(7):

$$\begin{aligned} L = \phi(x(t_1)) - \lambda^{k\prime}\left[h(x(t_1)) - \bar{h}^L\right] + \frac{\rho}{2}\left[h(x(t_1)) - \bar{h}^L\right]'\left[h(x(t_1)) - \bar{h}^L\right] - \\ - \int_{t_0}^{t_1} \mu^{k\prime}(t)\left[g(x(t), t) - \bar{g}^L\right] dt + \frac{\rho}{2}\int_{t_0}^{t_1} \left[g(x(t), t) - \bar{g}^L\right]'\left[g(x(t), t) - \bar{g}^L\right] dt - \\ - \int_{t_0}^{t_1} \gamma^k(t)\left[c(u, t) - \bar{c}^L\right] dt + \frac{\rho}{2}\int_{t_0}^{t_1} \left[c(u, t) - \bar{c}^L\right]'\left[c(u, t) - \bar{c}^L\right] dt, \end{aligned}$$

where $\bar{h}^L = h(x^k(t_1)) + h_x(x^k(t_1))\,\delta x(t_1)$, $\bar{g}^L = g(x^k(t), t) + g_x(x^k(t), t)\,\delta x(t)$, $\bar{c}^L = c(u^k(t), t) + c_u(u^k(t), t)\,\delta u(t)$, $\delta u = u - u^k$, $\delta x = x - x^k$.

Next, we linearize constraints (2), (3) at the k-th approximation:

$$I^k + \sum_{j=0}^{N} \nabla_u I^k(t^j)'(u_j - u_j^k) + \nabla_w I^{k\prime}(w - w^k) + \nabla_v I^{k\prime}(v - v^k) = 0,$$

$$J_j^k + \sum_{i=0}^{j}\Big[\nabla_u J^k(t^j)'(u_i - u_i^k) + \nabla_w J^k(t^j)'(w - w^k) + \nabla_v J^k(t^j)'(v - v^k)\Big] = 0, \quad j = \overline{0, N}.$$

Denoting by $A[m+(l+s)(N+1)] \times [r(N+1)+p+n]$ the matrix of coefficients of the linear equalities, by b — the vector of their free terms of dimension $m+(l+s)(N+1)$, and by Z — the vector of unknowns (u_j, $j = \overline{0, N}$; v; w) of dimension $r(N+1)+p+n$, we write the problem as follows:

$$L(z) \to \min, \qquad Az = b, \qquad z^L \le z \le z^U.$$

To solve the obtained linear programming problem, one can use numerical methods from the monographs [1,2].

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On the Topological Structure of Solution Set for System of Fractional Semilinear Inclusions with Hille–Yosida Operators in Banach Spaces

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In the present paper we prove that the set of all mild solutions to a system of fractional semilinear differential inclusions with Hille-Yosida operators in Banach spaces is an R_δ-set, i.e., it can be represented as the intersection of a decreasing sequence of compact contractible sets.

Keywords: fractional differential inclusion, system, mild solution, Hille-Yosida condition, measure of noncompactness, R_δ-set

Let $E_1, \dots, E_n$ be separable Banach spaces. We will consider the Banach space $E = E_1 \times \dots \times E_n$ equipped with the norm $\|x\|_E = \max_{1\le i\le n} \|x_i\|_{E_i}$, where $x = (x_1, \dots, x_n)$.

We will study the system of semilinear differential inclusions of fractional order in E:

$$\begin{cases} {}^C D_0^{q_1} x_1(t) \in A_1 x_1(t) + F_1(t, x_1(t), \dots, x_n(t)), & t \in [0,T], \\ \dots \\ {}^C D_0^{q_n} x_n(t) \in A_n x_n(t) + F_n(t, x_1(t), \dots, x_n(t)), & t \in [0,T], \end{cases} \tag{1}$$

where ${}^C D_0^{q_i}$ are Caputo derivatives of orders q_i, $0 < q_i < 1$, $i = 1, \dots, n$ (see following [1]); linear operators $A_i : D(A_i) \subseteq E_i \to E_i$, $i = 1, \dots, n$ satisfy the conditions:

(A1) each A_i is the Hille-Yosida operator, i.e., there exist constants $\omega_i \in \mathbb{R}$ and $C_i > 0$ such that $(\omega_i, +\infty) \subset \rho(A_i)$ and $\|\lambda I - A_i)^{-k}\|_{\mathcal{L}(E_i)} \le \frac{C_i}{(\lambda-\omega_i)^k}$ for all $\lambda > \omega_i$, $k \ge 1$.

For each $i = 1, \dots, n$, let $E_{0i} = \overline{D(A_i)}$ and A_{0i} be the part of the operator A_i defined as $A_{0i}x = A_i x \quad$ for $x \in D(A_{0i}) = \{x \in D(A_i) : Ax \in E_{0i}\}$.

It is known that each operator A_{0i} generates a C_0 - semigroup of linear operators $\{T_i(t)\}_{t\ge 0}$ on E_{0i}. We assume that

(A2) each semigroup $\{T_i(t)\}_{t\geq 0}$ is uniformly bounded, i.e., there exists $M_i \geq 1$ such that $\sup_{t\geq 0} \|T(t)\|_{\mathcal{L}(E_{0i})} \leq M_i$.

Define the multivalued nonlinearity $F : [0,T] \times E \to Kv(E)$ as $F(t,x) = F_1(t,x) \times ... \times F_n(t,x)$, where each multioperator $F_i : [0,T] \times E \to Kv(E_i)$, $i = 1, \dots, n$ satisfies the following conditions:

(F1) for each $x \in E$ the multifunction $F_i(\cdot, x) : [0,T] \to Kv(E_i)$ admits a measurable selection;
(F2) for a.e. $t \in [0,T]$, the multimap $F_i(t, \cdot) : E \to Kv(E_i)$ is u.s.c.;
(F3) there exists a function $\alpha_i(\cdot) \in L_+^{p_i}(0,T)$ with $p_i > \frac{1}{q_i}$ such that

$$\|F_i(t,x)\|_{E_i} \leq \alpha_i(t)(1 + \|x_i\|_{E_i}) \quad \text{a.e. } t \in [0,T].$$

Introduce the vector Hausdorff MNC $\mathcal{X}$ in the space E by setting, for a bounded set $\Omega \subset E$: $\mathcal{X}(\Omega) = \left(\chi_1(\Omega_1), \dots, \chi_n(\Omega_n)\right)^T \in \mathbb{R}_+^n$, where χ_i is the Hausdorff MNC in E_i and Ω_i denotes the projection of the space Ω onto E_i, $i = 1, \dots, n$. We will assume now that the multivalued nonlinearities F_i satisfy the condition (see[2]):

(F4) there exist functions $\mu_i \in L_+^{p_i}$ such that for each bounded set $\Omega \subset E$ we have $\mathcal{X}(F(t,\Omega)) \leq \mathcal{M}(t)\,\mathcal{X}(\Omega) \quad$ a.e. $t \in (0,T)$, where $\mathcal{M}(t)$ has the form:

$$M(t) = \begin{pmatrix} \mu_1(t) & 0 & \dots & 0 \\ 0 & \mu_2(t) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mu_n(t) \end{pmatrix}$$

We consider the problem on the existence of mild solutions $x(t) = (x_1(t), \dots, x_n(t))$ of system (1) satisfying initial conditions

$$x_1(0) = x_{01} \in E_{01}, \dots, x_n(0) = x_{0n} \in E_{0n}. \quad (2)$$

Definition 1. *A mild solution of system (1), (2) is the function $x(\cdot) \in C([0,\tau]; E)$, which can be represented in the form $x(t) = (x_1(t), ..., x_n(t))$, where $x_i(\cdot) \in C([0,\tau]; E_i)$, $i = 1, ..., n$,*

$$x_i(t) = \mathcal{G}_i(t)x_{0i} + \lim_{\lambda\to\infty} \int_0^t \mathcal{K}_i(t-s)B_i(\lambda)f_i(s)ds, \qquad \textit{with } f_i \in \mathcal{P}_{F_i}(x),$$

$$\begin{aligned}
&\mathcal{G}_i(t) = \frac{1}{\Gamma(1-q_i)} \int_0^t (t-s)^{-q_i}\mathcal{K}_i(s)s,\\
&\mathcal{K}_i(t) = t^{q_i-1}P_i(t),\\
&P_i(t) = q_i \int_0^\infty \theta\Psi_i(\theta)T_i(t^{q_i}\theta)d\theta,\\
&\Psi_i(\theta) = \sum_{k=0}^\infty \frac{(-\theta)^k}{k!\Gamma(-q_ik + 1 - q_i)},\\
&B_i(\lambda) = \lambda R(\lambda, A_i) = \lambda(\lambda I - A_i)^{-1}.
\end{aligned}$$

We can formulate the following main results for problem (1), (2) (see [3]).

Theorem 1. *Under conditions (A1), (A2), (F1) – (F4) the set of all mild solutions of system (1)–(2) is nonempty and compact.*

Theorem 2. *Under conditions (A1), (A2), (F1) – (F4) the set of all mild solutions of system (1)–(2) is an R_δ-subset of $\mathcal{C}$.*

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Quadratic Core of a Cooperative Game: Vertexes of the Recessive Cone Section*

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In the paper, we consider vertexes of the normalized section of recessive cone for so-called quadratic core of a finite cooperative game. This core was introduced by the author in 1978, and almost 20 years later it was rediscovered by French mathematician M. Grabish (named by him as 2-core). Quadratic core was elaborated in the initial article in order to replace classic core in case it is empty. The concept of nonadditive core assumes that payoff of the grand coalition may be utilized both for collective and individual assignment. Namely, part of this payoff is allowed to get as a public property to some "indivisible" two-player coalitions, while the rest of the payoff is given to the individual players as their private property. In the paper, we propose exhaustive description of the integer extreme vertexes of the normalized section of recessive cone of a TU cooperative game.

Keywords: cooperative game, Harsanyi dividends, quadratic core, recessive cone, vertexes of the normalized cone section

1 Main Results

Denote by $V(N)$ collection of cooperative n-player games, defined by their characteristic functions $v : 2^N \to \mathbb{R}$, where $N = \{1, \dots, n\}$ is player set, and for each coalition $S \subseteq N$ the value $v(S)$ treats as its maximal guaranteed payoff provided by cooperation of the members of S (remind that the vector space $V(N)$ was introduced by J. von Neumann in the fundemental book [1]). Given game $v \in V(N)$ and coalition S, denote by v_S Harsanyi dividend of S defined by the formula

$$v_S := \sum_{T \subseteq S} (-1)^{|ST|} v(T),\ S \subseteq N,$$

* The work was carried out in the framework of the State Task to Sobolev Institute of Mathematics (Project FWNF-2026-0022).

where symbol $|S\ T|$ denotes the number of elements of the set $S\ T$. Remind, that function $v \in V(N)$ is called ***quadratic*** if the following implication holds:

$$(|T| > 2) \Rightarrow (v_T = 0).$$

Subspace of quadratic set functions is denoted by $V_2(N)$.

One of the main optimal solutions of cooperative game theory is the core

$$C(v) := \{x \in \mathbb{R}^N \big| x(N) = v(N),\ x(S) \geq v(S),\ S \subseteq N\},$$

where $x(S) := \sum_{i \in S} x_i$. Since $C(v)$ is not empty for balanced games only, the problem of searching for an analog of classical core in non-balanced situation arises. In [2] a possible direction for elaboration of such analog was proposed. It rests on the exploiting of non-additive payoffs. Namely, in the approach worked out in [2], besides private, it was allowed apply collective property, as well. In more details, part of the collective payoff is allowed to get as a public property to some "indivisible" two-player coalitions, while the rest of the payoff is given to the individual players as their private property. To formalize this scheme we introduce the concept of ***unblockable quadratic imputation*** [2] of a game $v \in V(N)$ (that was rediscovered later in [3]): a set function $u \in V_2(N)$ is said to be unblockable provided that (i) $u(N) = v(N)$ and (ii) $u(S) \geq v(S)$, for each $S \subseteq N$. Respectively, the set

$$C_2(v) := \{u \in V_2(N) \big| u(N) = v(N),\ u(S) \geq v(S),\ S \subseteq N\}$$

of unblockable quadratic imputations of a game v is called its ***quadratic core***. It was established in [2] that unblockable quadratic imputations exist for each cooperative game $v \in V(N)$. But, in contrast to the classic (additive) core $C(v)$, quadratic core $C_2(v)$ is unbounded from above. It follows immediately from the fact that the recessive cone

$$A_2(N) := \{u \in V(N) \big| u(N) = 0,\ u(S) \geq 0,\ S \subseteq N\}$$

of the core $C_2(v)$ of any game $v \in V(N)$ contains nonzero elements. To investigate the cone $A_2(N)$ we consider the structure of its normalized section

$$B_2(N) := \{u \in A_2(N) \big| \sum_{i \in N} u(\{i\}) = 1\}.$$

Main result of the paper consists of the exhaustive description of the integer vertexes of the polyhedron $B_2(N)$.

Theorem *Collection of all the integer vertexes of polyhedron $B_2(N)$ consists of two series of quadratic set functions:* 1) $u^{(ij)},\ ij \in N^{(2)}$*; and* 2) $u^{(ijk)},\ ijk \in N^{(3)}$, *where*

$$u^{(ij)}(S) = \begin{cases} 1, \textit{if } i \in S \textit{ and } j \in N\ S, \\ 0\ \textit{otherwise}, \end{cases}$$

$$u^{(ijk)}(S) = \begin{cases} 1, \textit{if } S \cap ijk = i \textit{ or } S \cap ijk = jk, \\ 0\ \textit{otherwise}. \end{cases}$$

Here, we use the standard notations $N^{(2)} = \{T \subseteq N \big| |T| = 2\}$, $N^{(3)} = \{T \subseteq N \big| |T| = 3\}$, and $(ij)((ijk))$ is an ordered collection $(i,j)((i,j,k))$ consisting of the elements of the set $ij \in N^{(2)}(ijk \in N^{(3)})$.

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On Elliptic Equations in Cones with Different Smoothness*

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A model elliptic pseudo-differential equation is considered in a special multi-dimensional cone. Using a special factorization for an elliptic symbol a general solution for the equation is described.

Keywords: elliptic pseudo-differential equation, cone, wave factorization, solvability

1 The main results

Let be the Euclidean space $\mathbb{R}^m$ is represented as an orthogonal sum

$$\mathbb{R}^M = \mathbb{R}^{k_1} \oplus \dots \oplus \mathbb{R}^{k_n},$$

taking into account that the sets K_j form a partition of M, $k_j = card\ K_j$. For the vector $S = (s_1, \dots, s_n)$, the space $H^S(\mathbb{R}^M)$ is defined as the space of distributions with the finite norm

$$||u||_S^2 = \int_{\mathbb{R}^M} \prod_{j=1}^n (1 + |\xi_{K_j}|)^{2s_j} |\tilde{u}(\xi)|^2 \, d\xi,$$

where $\tilde{u}$ denotes the Fourier transform of u [1].

The pseudo-differentiated operator under consideration is the following

$$(Au)(x) = \frac{1}{(2\pi)^M} \int_{\mathbb{R}^M} \int_{\mathbb{R}^M} e^{i(x-y)\cdot\xi} \tilde{A}(\xi) u(y) \, dy \, d\xi,$$

whose symbol satisfies the condition

$$c_1 \prod_{j=1}^n (1 + |\xi_{K_j}|)^{\alpha_j} \le |\tilde{A}(\xi)| \le c_2 \prod_{j=1}^n (1 + |\xi_{K_j}|)^{\alpha_j}.$$

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The main focus is on the study of the equation

$$Au = 0 \tag{1}$$

in the cone $C = C_{K_1} \times C_{K_n}$, where $C_{K_j} \subset \mathbb{R}^{k_j}$ is a sharp convex cone.

The space $H^S(C)$ is defined as a subspace $H^S(\mathbb{R}^M)$, consisting of functions whose supports are contained in $\overline{C}$.

Wave factorization plays a key role in this approach

$$\tilde{A}(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

it is a special representation for an elliptic symbol, it has a certain index $\kappa = (\kappa_1, \dots, \kappa_n)$ [2,3].

Let the symbol $\tilde{A}(\xi)$ admits wave factorization with respect to the convex cone C with the index $\kappa = (\kappa_1, \dots, \kappa_n)$. If you have the conditions

$$|\kappa_j - s_j| < 1/2, \qquad j = 1, \dots, n,$$

this equation (1) in the space $H^S(C)$ has only a trivial solution.

If the ratio is met $\kappa - S = N + \varepsilon$, $N = (n_1, \dots, n_n) \in \mathbb{N}^n$, $|\varepsilon_j| < \frac{1}{2}$, then the general solution of equation (1) in the space $H^S(C)$ in Fourier images has the form

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\, V_{\varphi}^{-1} \left(\sum_{l_1=1}^{n_1} \cdots \sum_{l_n=1}^{n_n} \tilde{c}_L(\xi'_K)\, \xi_{k_1}^{l_1-1} \cdots \xi_{k_n}^{l_n-1} \right), \tag{2}$$

where $\tilde{c}_L \in \tilde{H}^{S_L}$ are arbitrary functions.

$$S_L = (s_1 - æ_1 + l_1 - 1/2, \dots, s_n - æ_n + l_n - 1/2).$$

The a priori estimate

$$||u||_S \leq const \sum_{l_1=1}^{n_1} \sum_{l_2=1}^{n_2} \cdots \sum_{l_n=1}^{n_n} ||c_L||_{S_L}.$$

holds.

The formula (2) includes the special operator V_{φ} which is constructed for each cone separately.

The obtained representation shows that in these cases a finite-dimensional solution space arises, the structure of which is determined by the wave factorization index. To identify a single solution, additional conditions must be set. In the simplest geometric cases , the corresponding boundary value problems are reduced to systems of linear integral equations [2].

Thus, consideration of elliptic pseudodifferential equations in spaces of different smoothness with respect to variables makes it possible to describe the structure of solutions and correctly formulate boundary value problems. in non-smooth areas.

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Comparative Analysis of SLAM Algorithms for Mobile Robots

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A fundamental task in the field of mobile robotics is to ensure reliable autonomous navigation in unknown environments. This task becomes critical in indoor spaces, underground mines, and dense urban areas where satellite signals are unavailable or subject to severe distortion. The solution to this problem is the Simultaneous Localization and Mapping (SLAM) technology, which is a process of continuous computer modeling of the environment based on noisy sensor data while simultaneously estimating the robot's own trajectory.

Keywords: SLAM, mobile robotics, mathematical modeling, graph optimization, LiDAR, particle filter, autonomous navigation

1 Introduction and Mathematical Background

From a mathematical point of view, the fundamental SLAM problem is reduced to the problem of recursive Bayesian estimation. The state of the system is described by two components: the robot's position vector x_k and a set of landmarks on the map m. The robot motion model describes changes only in the position of the robot in accordance with the control actions u_k, leaving the map states unchanged, which is formally expressed by the dynamics equation:

$$x_k = f(x_{k-1}, u_k) + \omega_k,$$

where ω_k is the process noise.

Historically, probabilistic filters were the first mathematical tools to solve this problem. The classical approach based on the Extended Kalman Filter (EKF-SLAM) [1] models system uncertainty through covariance matrices. However, as the map grows, the computational complexity of EKF-SLAM increases quadratically with respect to the number of landmarks, making the model inapplicable for large-scale spaces. The subsequently developed FastSLAM algorithm [2] allowed decoupling the localization and mapping tasks using a particle filter (Monte Carlo methods), partially solving the data association problem but retaining a high computational load.

2 Graph Optimization and Algorithm Analysis

The modern de facto standard is the transition from filtering to the graph optimization paradigm (Graph-SLAM) [3]. In this mathematical model, the problem is represented as a stochastic graph, where the nodes are the historical positions of the robot, and the edges are spatial constraints (sensor measurements and odometry). State estimation is reduced to minimizing the residual (error) function using nonlinear optimization algorithms, such as the Gauss-Newton or Levenberg-Marquardt methods. The use of sparse data structures and the division of the global map into local submaps allowed the algorithms to function in real time.

Based on graph and probabilistic models, a comparative analysis of key algorithms using 2D-LiDAR data was conducted:

1. **GMapping** [4] – an algorithm based on a particle filter with adaptive resampling. It demonstrates high accuracy in small areas but requires significant amounts of RAM when scaling.
2. **Hector SLAM** [5] – utilizes scan-matching optimization based on high-frequency LiDAR data, allowing the construction of a spatial model without using wheel odometry data.
3. **Cartographer** [6] – a modern solution from Google that uses graph optimization and the concept of submaps. It provides real-time loop closure and supports integration with inertial measurement units (IMU).

3 Limitations and Future Directions

Despite the high accuracy of modern graph models, the analysis revealed a number of critical limitations when applied in real-world conditions. Firstly, the mathematical apparatus of 2D-LiDAR SLAM is highly sensitive to weather artifacts: raindrops or snow are modeled by the system as static obstacles, which leads to distortion of the map topology (“phantom” walls). Secondly, classical algorithms assume the surrounding world is static. The presence of dynamic objects (moving people, vehicles) introduces significant perturbations into the graph optimization process. Thirdly, in conditions of geometric degeneracy (e.g., in long corridors without distinctive features), the algorithms lack data for correct scan matching, leading to unpredictable model behavior.

Thus, the evolution of SLAM algorithms demonstrates a successful transition from computationally heavy probabilistic filters to fast graph optimization methods. However, ensuring reliable navigation in complex environments requires the development of modified algorithms. Currently many researches are focused on creating hybrid multi-sensor systems (Sensor Fusion), integrating 3D-LiDAR (LOAM, FAST-LIO2 algorithms), and applying deep learning methods for semantic filtering of dynamic disturbances and weather artifacts at the stage preceding the construction of the mathematical graph. A separate promising area of computer modeling is collaborative SLAM (C-SLAM), which solves the problem of distributed construction of a globally consistent map by a group of robots.

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Fault-Tolerance Control Based on Integral Super-Twisting Algorithm and Barrier Function for Quadrotors

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This paper addresses the problem of quadrotor control under unknown external disturbances and actuator failures. A passive fault-tolerant control strategy is proposed that combines sliding mode control (SMC), the super-twisting algorithm (STA), and a barrier function. The stability of the closed-loop system is proven using Lyapunov theory. The simulation results confirm the superiority of the proposed method over its counterparts in terms of trajectory tracking accuracy and convergence speed.

Keywords: quadrotor, sliding mode control, fault tolerance, super-twisting algorithm, barrier function

1 Introduction

Quadrotors are widely used in civilian and military applications due to their maneuverability and compactness. However, their control is complicated by nonlinear dynamics, underactuation, and sensitivity to external disturbances and motor failures. Traditional sliding mode control (SMC) methods require knowledge of the upper bounds of disturbances, which is difficult to implement in practice. Adaptive methods using neural networks or observers increase computational complexity. The goal of this work is to develop a computationally efficient controller that ensures stability under unknown failures without using complex estimators.

2 Control Method

The quadrotor dynamics are described by Newton-Euler equations and decomposed into positional and angular subsystems. Actuator failures are modeled as additive disturbances with unknown coefficients.

The proposed controller combines equivalent control and a switching component based on the super-twisting algorithm (STA):

$$u = u_{eq} - \beta \|S\|^{1/2} \operatorname{sign}(S) - \int \beta^2 \operatorname{sign}(S)\, dt, \tag{1}$$

where S is the sliding surface, and β is the gain coefficient determined by the barrier function:

$$\beta(s) = \frac{\sqrt{\varepsilon}b}{(\varepsilon - \|s\|)^{1/2}}. \tag{2}$$

The barrier function enables adaptive gain adjustment depending on the deviation from the sliding surface, compensating for disturbances without knowledge of their bounds. This eliminates the parameter mismatch problem and reduces the chattering effect. System stability is guaranteed in finite time, as proven via a Lyapunov function.

3 Simulation Results

The effectiveness of the method was verified in Google Colab on a spiral trajectory tracking task. An actuator failure was simulated at the 10th second of flight. Comparison was performed with methods from [1] and [2].

Key results:

- **Accuracy:** The proposed method ensures lower tracking error. Position RMSE is reduced by 28.26%, and attitude RMSE by 17.81% compared to [3].
- **Convergence speed:** System convergence time is reduced by 0.54 s compared to [3].
- **Robustness:** After failure occurrence, no significant trajectory fluctuations are observed, unlike the comparison methods.

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An Algorithm for Finding Suboptimal Configurations in the Generalized Min-Max Ratio Problem

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We consider a version of the min-max ratio problem related to the Borsuk conjecture on partitioning sets into parts of smaller diameter and the Gale conjecture on partitions of a sphere. Algorithms for finding suboptimal configurations in this problem are proposed.

Keywords: distance geometry, Borsuk problem, chromatic number

The problem of minimizing the ratio of the maximum distance to the minimum distance in the Euclidean space was first considered by Erdős and Bateman in [1]. For a given dimension n and number of points m, one have to find

$$r_{n,m} = \sup_{v_1,\ldots,v_m \in \mathbb{R}^n} \frac{\max_{i<j} \|v_i - v_j\|}{\min_{i<j} \|v_i - v_j\|}.$$

Note that for any fixed values of the parameters, the problem reduces to optimization over a compact set, so the supremum is attained at some specific set of points.

In the talk, we will consider a generalization of this problem in which we need to maximize the ratio of the diameter of a finite set $V = \{v_1, \ldots, v_m\}$ to the minimax of the diameters of k disjoint subsets $V_1, \ldots, V_k$ into which the original set is partitioned.

$$r_{n,m,k} = \sup_{v_1,\ldots,v_m \in \mathbb{R}^n} \frac{\mathrm{diamV}}{\min_{V_1,\ldots,V_k} \max \operatorname{diam} V_k}, .$$

In other words, we seek a finite set that is the “hardest” to partition into k parts in order to reduce the diameter. This problem is directly related to the following two conjectures.

The Borsuk conjecture: any compact subset of $\mathbb{R}^n$ of positive diameter can be partitioned into $n+1$ parts of strictly smaller diameter. (It holds in dimensions 1, 2, and 3, and is false in dimensions 64 and above.)

The Gale conjecture: when partitioning a compact subset of $\mathbb{R}^n$ into $n+1$ parts, one can achieve a ratio of the diameter of the set to the maximum diameter of the parts that is no larger than that achieved by partitioning a ball. (It holds in dimensions 1 and 2, and is false in dimensions 64 and above.)

Problems of this type are of independent interest as a benchmark for the development of scientific research support tools based on large language models [2,3].

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Gauge-Controlled Obstruction to Kerr Hidden Integrability

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We study the perturbative survival of hidden integrability in weakly deformed Kerr geometry. Although stationarity and axisymmetry protect the energy E and axial angular momentum L_z, complete Kerr integrability also requires the hidden Killing–Yano structure generating the Carter invariant. We formulate its loss as a gauge-controlled cokernel obstruction. A stationary-axisymmetric quadrupolar bump preserves E and L_z, but generically induces Carter drift and a deformation of the EMRI frequency map.

Keywords: Kerr geometry, hidden symmetry, Killing–Yano tensor, Carter invariant, EMRI

1 The main results

The Kerr metric is special not only because it is stationary and axisymmetric, but because it possesses a hidden Killing–Yano tensor. Its square gives the Killing tensor responsible for the Carter invariant, making generic bound geodesic motion completely integrable. Therefore, in precision tests of the Kerr hypothesis, especially with extreme-mass-ratio inspirals, conservation of the manifest charges E and L_z is not enough: one must also test whether the hidden separability structure survives. This perspective is closely connected with nonlinear field dynamics and spacetime self-organization in Kerr geometry [1].

Let a weak perturbation of Kerr be written as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \varepsilon h_{\mu\nu} + O(\varepsilon^2), \qquad 0 < \varepsilon \ll 1. \tag{1}$$

The manifest symmetry sector is governed by the Lie derivatives $\mathcal{L}_{\xi_{(t)}} h_{\mu\nu}$ and $\mathcal{L}_{\xi_{(\varphi)}} h_{\mu\nu}$. If they vanish, the perturbed Hamiltonian remains stationary and axisymmetric, and E, L_z are conserved. The hidden sector is more restrictive. For a deformed Killing–Yano tensor $\tilde{Y}_{\mu\nu} = Y_{\mu\nu} + \varepsilon \delta Y_{\mu\nu} + O(\varepsilon^2)$, the first-order Killing–Yano equation has the schematic form

$$D_{\mathrm{KY}} \delta Y = J[h, Y]. \tag{2}$$

A regular continuation exists only if $J[h, Y] \in \operatorname{Im} D_{\mathrm{KY}}$. However, under a linearized diffeomorphism, $h_{\mu\nu} \mapsto h_{\mu\nu} + \mathcal{L}_\eta g^{(0)}_{\mu\nu}$, the source changes by an image term, $J \mapsto J + D_{\mathrm{KY}}(\mathcal{L}_\eta Y)$. Hence only the cokernel projection is physical. On an orbital tube U, with fixed boundary conditions, the gauge-controlled obstruction is

$$O_{\mathrm{KY}}[h] = \left[\sum_i |\langle Z_i, J[h, Y] \rangle_U|^2 \right]^{1/2}, \qquad Z_i \in \ker D^\dagger_{\mathrm{KY}}.$$

If $O_{\mathrm{KY}}[h] \neq 0$, the Kerr Killing–Yano structure cannot be continued in the chosen physical class.

As a diagnostic example, consider a stationary-axisymmetric quadrupolar bump of the inverse Kerr metric,

$$\delta g^{tt} = -\Psi g^{tt}_K, \qquad \delta g^{rr} = -\Psi g^{rr}_K, \qquad \Psi(r, \theta) = \left(\frac{M}{r}\right)^3 P_2(\cos\theta).$$

This perturbation is independent of t and φ; therefore E and L_z remain conserved. Nevertheless, the first-order Hamiltonian perturbation is

$$H_1 = -\frac{1}{2}\Psi(r, \theta) \left(g^{tt}_K E^2 + g^{rr}_K p_r^2 \right).$$

Kerr separability requires ΣH, with $\Sigma = r^2 + a^2 \cos^2\theta$, to split into radial and polar parts. For the bump above, $\partial_r \partial_{\cos\theta}(\Sigma H_1) \neq 0$ for generic rotating, inclined, and noncircular orbits. Thus stationarity and axisymmetry do not guarantee Carter–Staeckel separability.

The same result appears dynamically through the Carter invariant:

$$\frac{dQ_K}{d\tau} = \varepsilon p_\theta \partial_\theta \left[\Psi\left(g_K^{tt} E^2 + g_K^{rr} p_r^2\right)\right] + O(\varepsilon^2). \tag{3}$$

Hence, for generic off-equatorial motion,

$$\dot{E} = 0, \qquad \dot{L}_z = 0, \qquad \dot{Q}_K \neq 0.$$

For EMRI dynamics this produces a deformation of the Kerr frequency map. In action–angle variables, $H(J, q) = H_0(J) + \varepsilon H_1(J, q) + O(\varepsilon^2)$, and on a nonresonant reference torus

$$\delta\Omega_i = \varepsilon \frac{\partial}{\partial J_i} \langle H_1 \rangle_J + O(\varepsilon^2).$$

Thus $\dot{E} = 0$ and $\dot{L}_z = 0$ do not imply $\delta\Omega_i = 0$. The hidden-symmetry obstruction is therefore a distinct strong-field channel by which weak non-Kerr structure can enter waveform phases. The relevant diagnostic is the gauge-controlled projection $P_{\mathrm{coker}}(D_{\mathrm{KY}})J[h, Y]$, rather than the norm of $h_{\mu\nu}$ alone.

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Numerical Method for Solving the Inverse Heat Conduction Problem under Uncertainty

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The paper is devoted to numerical solving inverse heat conduction problem with incomplete initial data. The mathematical representation of problem includes a one-dimensional parabolic PDE, the initial condition and single boundary condition. We propose the approach based on the regularization technique and allow to obtain the numerical solution under uncertainty. This approach was implemented in numerical scheme and tested due computational experiments. The obtained results show the sufficient accuracy of the proposed scheme.

Keywords: computational scheme, inverse heat conduction problem, regularization

1 The main results

Data processing of current thermal state of the object provides an important component to success in resolving the optimization issue for technological processes. The basis of the information are the results of the boundary temperature measurements. The issue in which internal temperatures are calculated from indirect data belongs to the inverse problem. The creation of high-quality data processing methods that allow one to solve the inverse problem is a topical area of research [1,2,3,4,5,6,7,8].

We consider the following inverse problem:

$$u_t = au_{xx}, \quad (x,t) \in (0,L) \times (0,T), \tag{1}$$

where a is the reduced coefficient. The initial thermal state of the object is represented by the following condition:

$$u(x,0) = f(x), \quad x \in [0,L]. \tag{2}$$

In respect with the sensors location we can measure the temperature on the one boundary part only. Thus, the boundary condition is formulated as follows:

$$u(0,t) = p(t), \quad t \in [0,T]. \tag{3}$$

Based on the specific features of the technological process, we can presume that

$$u(x,t) \in H^{2+\gamma,1+\gamma/2}\left((0,L) \times (0,T)\right) \cap H^{1,\gamma/2}([0,L] \times [0,T]),\ \gamma \in (0,1),$$

and

$$f(x) \in H^1([0,L]),\ p(t) \in H^{\gamma/2}([0,T])$$

for any $T > 0$. The inverse problem is to determine $u(x,t)$ in the domain $(x,t) \in [0,L] \times [0,T]$ and to find $u(L,t) = \psi(t)$.

The specificity of applied mathematical problems related to technological processes is noise in result measurements that inevitably leads to the errors in initial data. In this study, the situation where the initial data contain the noise is represented as follows. Instead of the exact values $p(t)$ the approximations $p_\delta(t)$ and the allowable noise level δ, such that

$$\|p(t) - p_\delta(t)\| \le \delta$$

are given. The common difficulty in numerically solving the inverse problems is the responsiveness of computational procedures to initial data errors, when a low level of initial data errors leads to significant distortion in data processing results. To ensure the stability of the computational scheme, numerical regularization is used.

In this work, we propose an approach to create a computational scheme for solving the heat conduction inverse problem (1)–(3) under uncertainty. The mathematical representation of issue involved the one-dimensional parabolic PDE, initial and single boundary conditions. The absently second boundary condition leads to uncertainty. The proposed numerical method is based on finite-difference equations. The stability of the proposed computational scheme is ensured by regularizing technique and confirmed by results of computational experiments, including a comparative analysis of numerical solutions with test values.

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Minimum Residual Parallel Multisplitting Iterative Method for Systems of Linear Equations

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In this paper we present a minimum residual parallel multisplitting iterative method(MRPM) which uses the minimum residual technique at each iteration step for solving a system of linear equations. We prove that the MRPM iteration method is convergent and show the feasibility and effectiveness of the method by a numerical example.

Keywords: parallel multisplitting, minimum residual, iterative method, convergence

1 Introduction

Many problems in engineering and practice lead to the solution of a system of linear equations

$$Ax = b, \quad A = (a_{ij}) \in C^{n\times n},\ b, x, \in C^n \tag{1}$$

where A is the coefficient matrix, b is the right side vector, and x is the unknown vector. The parallel multisplitting iterative method for the solution of linear systems can be written as follows:

$$\begin{aligned} &A = M_i - N_i, \quad i = 1, 2, \cdots m \\ &M_i x_i^{(k)} = N_i x_i^{(k-1)} + b, \quad i = 1, 2, \cdots \\ &x^{(k)} = \sum_{i=1}^{m} E_i x_i^{(k)} \end{aligned} \tag{2}$$

There has been a lot of study on the parallel multisplitting iterative method for the cases that A is an H-matrix [1] and an Hermitian positive definite matrix [2]. In this paper, we focus on an acceleration of the parallel multisplitting iteration method by using the minimum residual technique ([3]) at each step of the iteration for solving the systems of linear equations.

2 Minimal residual parallel multisplitting iteration method

The algorithm of the MRPM iteration method is as follows.
Step 1. Compute in parallel

$$x_i^{(k)} = x^{(k-1)} + \omega_i^{(k-1)} \delta_i^{(k-1)}, \ i = 1, 2, \cdots m$$

where $\omega_i^{(k-1)} = \frac{(r^{(k-1)}, A\delta_i^{(k-1)})}{\|A\delta_i^{(k-1)}\|}$, $\delta_i^{(k-1)} = M_i^{-1} r^{(k-1)}$.
Step 2. Compute

$$x^{(k)} = \sum_{i=1}^{m} \alpha_i^k x_i^{(k)}, \ k = 1, 2, \cdots$$

where $\alpha^{(k)} = (\alpha_1^{(k)}, \alpha_2^{(k)}, \cdots, \alpha_m^{(k)})$ is the solution to the following quadratic programming

$$\min_{\alpha} \frac{1}{2} r^* r, \quad r = A\Big(\sum_{i=1}^{m} \alpha_i^k x_i^{(k)}\Big) - b \tag{3}$$

Step 3. If $\|r^{(k)}\| < \varepsilon$, stop; Otherwise, $k \Longleftarrow k+1$ and go back to Step 1.

3 Analysis of convergence

Theorem. Let M_1 be satisfied $(M_1\delta, A\delta) \neq 0$ for any nonzero vector $\delta \in C^n$. Then the MRPM iteration method converges to the solution of (1) and the convergent rate is given

$$q = -\ln\sqrt{1 - \frac{d^2}{\|AM_1^{-1}\|^2}}.$$

4 Numerical experiments

We consider the convection-diffusion equation

$$\begin{cases} -Pe^{-1} \ \ u + \frac{1}{2}\Big[v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y} + \frac{\partial v_1 u}{\partial x} + \frac{\partial v_2 u}{\partial y}\Big] = g, \quad \Omega \\ u = 0, \quad \partial\Omega \end{cases} \tag{4}$$

where $\Omega = (0,1) \times (0,1)$ and $v_i = v_i(x,y)(i = 1,2)$ to be the continuous functions listed in Table 1. We listed the iteration numbers of our method and Algorithm 2.3 in [2] in Table 2.

Table 1: The test problems.

$Prob.no.$	$v_1(x,y)$	$v_2(x,y)$
1	1	-1
2	$1-2x$	$2y-1$
3	$x+y$	$x-y$

Table 2: Performance comparison of the algorithms.

$Prob.no.$	$Algorithm2.3$	$OurAlgorithm$
$Prob.1$	938	493
$Prob.2$	581	323
$Prob.3$	646	448

5 Conclusion

In this paper we proposed a parallel multisplitting iterative method which uses the minimum residual technique at each iteration step for solving a system of linear equations and prove the convergence. A numerical example shows the effectiveness of our method.

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Implementing Multicriteria Optimization Method via Cutting Procedures

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We propose a variant of the classical successive concessions method for solving multi-objective optimization problems. The primary distinction of our approach lies in the specific technique used to define the concessions. This method allows us to obtain specified deviations between solutions of adjacent stages not only by the value of the objective functions, but also by the distance. We describe one of the possible implementations of the proposed variant by using cutting procedures when solving particular problems.

Keywords: multi-objective optimization, cutting methods, successive concessions method

The m-criterial problem (e.g., [1,2]) is solved, where the criteria are defined by functions $f_j(x)$, $j = 1, \dots, m$, which are convex on $\mathrm{R_n}$, and the constraint set $D \subset \mathrm{R_n}$ is convex, closed, and bounded with $\operatorname{int} D \neq \emptyset$. We will assume that the criteria are numbered in descending order of their importance, and the particular problems have the form $\min\{f_j(x) : x \in D\}$ for all j. The proposed variant of the successive concessions method consists of m stages and is as follows.

Select the numbers ε_k, $\delta_k > 0$, $k = 1, \dots, m-1$. Assume $D_1 = D$, $k = 1$.

1^0. The point $x_k = (\xi_1^k, \dots, \xi_n^k) \in D_k$ is found as a solution to the problem

$$\min\{f_k(x) : x \in D_k\}. \tag{1}$$

If $k = m$, then the point x_k is taken as the solution x^* of the original problem.

2^0. Construct the sets $G_k = \{x \in \mathrm{R_n} : f_k(x) \leq f_k(x_k) + \varepsilon_k\}$, $U_k = \{x = (\xi_1, \dots, \xi_n) \in \mathrm{R_n} : \xi_i^k - \delta_k \leq \xi_i \leq \xi_i^k + \delta_k,\ i = 1, \dots, n\}$, $D_{k+1} = D_k \cap G_k \cap U_k$, and go to Step 1^0 with k increased by one.

Note that the sets U_k can also be defined in other ways at Step 2^0. Moreover, the selection of ε_k, δ_k allows to exclude G_k or U_k during constructing D_{k+1} for all or some k.

If the functions $f_j(x)$, $j \in J$, are linear and D is a polyhedron, then the set D_k is also a convex polyhedron, and problem (1) can be solved efficiently. Therefore, we will assume that convex programming problem (1) has a general form for all k. Taking this into account, as an implementation of the described method, a cutting algorithm is proposed below that uses an approximation of both the set D_k and the epigraph of the function $f_k(x)$ for each k, and is characterized by the construction of points within D_k. This algorithm is as follows.

Construct the convex compact set $U_0 \subset \mathrm{R_n}$ such that $D \subset U_0$. Define the numbers ε_k, $\delta_k > 0$, $k = 1, \dots, m-1$, $\Delta_k > 0$, $k = 1, \dots, m$, $q > 1$. Put $D_1 = D$, $k = 1$.

1. Let $Q_k^0 = U_{k-1}$. Construct the convex closed set $M_k^0 \subset \mathrm{R_{n+1}}$ such that $\operatorname{epi}(f_k, \mathrm{R_n}) \subset M_k^0$ and the inequality $\gamma \geq \bar{\gamma}_k$ holds for all $(x, \gamma) \in M_k^0$, where $x \in U_{k-1}$, $-\infty < \bar{\gamma}_k \leq \min\{f_k(x) : x \in U_{k-1}\}$. Choose $v_k \in \operatorname{int} D_k$ and put $v_k' = (v_k, \theta_k)$, where $\theta_k > f_k(v_k)$, $w_k^{-1} = v_k$, $i = 0$.

2. Find the solution $u_k^i = (y_k^i, \gamma_k^i)$ to the problem $\min\{\gamma : (x, \gamma) \in M_k^i, x \in Q_k^i\}$. If $y_k^i \in D_k$ and $f_k(y_k^i) = \gamma_k^i$, then y_k^i is a solution to problem (1), assign $x_k = y_k^i$ and go to Step 7.

3. If $y_k^i \in D_k$, then fix $z_k^i = y_k^i$, $\tilde{y}_k^i = y_k^i$. Otherwise, $z_k^i = \lambda_k^i v_k + (1 - \lambda_k^i) y_k^i$, $\tilde{y}_k^i = y_k^i + q_k^i (z_k^i - y_k^i)$, where $\lambda_k^i \in (0, 1)$, $q_k^i \in [1, q]$, such that $z_k^i \notin \operatorname{int} D_k$, $\tilde{y}_k^i \in D_k$. If $f(\tilde{y}_k^i) = \gamma_k^i$, then $\tilde{y}_k^i$ is a solution to problem (1), assign $x_k = \tilde{y}_k^i$ and go to Step 7.

4. Choose the point $w_k^i \in D_k$ satisfying the condition $f_k(w_k^i) \leq \min\{f_k(\tilde{y}_k^i), f_k(w_k^{i-1})\}$. If $f_k(w_k^i) = \gamma_k^i$, then w_k^i is a solution to problem (1), and if $f_k(w_k^i) - \gamma_k^i \leq \Delta_k$, then w_k^i is a Δ_k-solution to problem (1). In both cases, it is assumed that $x_k = w_k^i$ and proceed to Step 7.

5. Construct the set Q_k^{i+1}. If $y_k^i \in D$, then $Q_k^{i+1} = Q_k^i$. Otherwise, $Q_k^{i+1} = Q_k^i \cap \{x \in \mathrm{R_n} : \langle a_k^i, x - z_k^i \rangle \leq 0\}$, where a_k^i is a support vector for D_k at the point z_k^i and $\|a_k^i\| = 1$.

6. Construct the set M_k^{i+1} as follows. Choose $r_k^i = \alpha_k^i v_k' + (1 - \alpha_k^i) u_k^i$, where $\alpha_k^i \in (0, 1)$ is constructed according to $r_k^i \notin \operatorname{int} \operatorname{epi}(f_k, \mathrm{R_n})$ and $u_k^i + \tilde{q}_k^i (r_k^i - u_k^i) \in \operatorname{epi}(f_k, \mathrm{R_n})$ holds for some $\tilde{q}_k^i \in [1, q]$. Put $M_k^{i+1} = M_k^i \cap \{u \in \mathrm{R_{n+1}} : \langle b_k^i, u - r_k^i \rangle \leq 0\}$, where b_k^i is a support vector for $\operatorname{epi}(f_k, \mathrm{R_n})$ at the point r_k^i and $\|b_k^i\| = 1$. After incrementing i by 1, proceed to Step 2.

7. If $k = m$, then $x^* = x_k$, and the process is completed. Otherwise, construct the sets G_k, U_k, D_{k+1} according to Step 2^0 of the proposed method, and proceed to Step 1 under $k := k + 1$.

We validate the optimality and Δ_k-optimality criteria defined at Steps 2–4. For every k, we identify a specific $i = i_k$ that yields a fixed Δ_k-solution $x_k = w_k^{i_k}$ for the k-stage problem.

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Optimal Control in a Degenerate Hydrodynamic Model with Cubic Nonlinearity and Damping*

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The existence of a unique solution to the initial-boundary value problem for a nonhomogeneous equation of a second-order Sobolev-type semilinear equation is proven. The extendability of the local solution over an arbitrary time interval is demonstrated, allowing the formulation and solution of an optimal control problem. In the optimal control problem, the optimality criterion is the balance (classical) functional; the norms used in it are determined by the existence theorem for a solution to the initial-boundary value problem. As an example, a mathematical model of the propagation of decaying waves in shallow water with cubic nonlinearity is considered.

Keywords: Sobolev-type semilinear equations, solution continuation, optimal control, balance functional, Galerkin method

The study of nonlinear wave equations plays a crucial role in understanding various physical phenomena in various scientific and engineering disciplines. One such equation is an improved version of the classical Boussinesq equation and describes the propagation of nonlinear long waves in shallow water (e.g., tsunamis). In real physical systems, wave energy dissipates. When dealing with waves in liquids or viscoelastic media, a term describing hydrodynamic attenuation [1] is introduced into the equation. Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary $\partial\Omega$ of class C^∞, $T \in \mathbb{R}_+$. In the cylinder $\Omega \times (0, T)$, we consider the Boussinesq–Love equation

$$(\lambda - \Delta)x_{tt} + \alpha^2(\lambda_1 - \Delta)x_t + \beta^2(\lambda_2 - \Delta)x + (x)^3 = u(s,t) \tag{1}$$

with the homogeneous Dirichlet boundary condition

$$x(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0,T), \tag{2}$$

and the Showalter–Sidorov initial conditions

$$P(x(s,0) - x_0(s)) = 0, \quad P(x_t(s,0) - x_1(s)) = 0, \quad s \in \Omega, \tag{3}$$

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where $\lambda, \lambda_1, \lambda_2, \alpha, \beta \in \mathbb{R}$, Δ is the Laplace operator, $u(s,t)$ is the heterogeneity (control) function.

Using the theory of relatively polynomially bounded operator pencils developed by A.A. Zamyshlyaeva [2], it is shown in that suitably chosen spaces, the (1)–(3) problem can be reduced to an abstract semilinear second-order Sobolev-type equation

$$A\ddot{x} + B_1\dot{x} + B_0 x + (x)^3 = u, \tag{4}$$

$$P(x(0) - x_0) = 0, \quad P(\dot{x}(0) - x_1) = 0, \tag{5}$$

where A, B_1, B_0 are linear and continuous operators acting from the Banach space $\mathfrak{U}$ to the Banach space $\mathfrak{F}$, $\ker A \neq \{0\}$.

First, we solve problem (4), (5) using the Galerkin method and compactness [3]. To do this, we construct several function spaces. Let $L^2 = (L^2, \langle \cdot, \cdot \rangle)$ be a real, separable, Hilbert space. We define dual pairs of reflexive Banach spaces $(H^{1,0}, H^{-1})$ and $(L^4, L^{4/3})$ with respect to the duality $\langle \cdot, \cdot \rangle$.

We will impose conditions on operators A, B_1, B_0:
(S1) $A \in (H^{1,0}, H^{-1})$ – self-adjoint, non-negative definite, Fredholm;
(S2) $B_0, B_1 \in (H^{1,0}, H^{-1})$ – self-adjoint, non-negative definite.

Theorem 1. *[4] Let $x_0 \in \mathfrak{A} \cap L^4$, $x_1 \in H^{1,0}$, $u \in L^{4/3}(0,T; H^{-1})$ and conditions **(S1)**, **(S2)** hold. Then there exists a unique solution to problem (4), (5) $x = x(s,t)$ such that $x \in L^\infty(0,T; H^{1,0} \cap L^4)$ and $\dot{x} \in L^\infty(0,T; H^{-1})$.*

We consider the optimal control problem (6)

$$\begin{gathered} J(x,u) = \beta(\|x(t) - z(t)\|^4_{L^4(0,T;L^4)} + \\ + \|\dot{x}(t) - \dot{z}(t)\|^2_{L^2(0,T;L^2)}) + (1-\beta)\|u(t)\|^2_{L^2(0,T;H^{-1})} \to \min \end{gathered} \tag{6}$$

To do this, we construct the space $\mathfrak{U} = L^2(0,T; L^2)$ and define a nonempty closed and convex set $\mathfrak{U}_{ad}$ in it. Based on Theorem 1, we construct the space $\mathfrak{X}_1 = \{x | x \in L^\infty(0,T; H^1_0 \cap L^4), \dot{x} \in L^\infty(0,T; L^2)$.

Definition 1. *A pair $(\tilde{x}, \tilde{u}) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$ is called a solution to the optimal control problem if*

$$J(\tilde{x}, \tilde{u}) = \inf_{(x,u)} J(x,u),$$

where the pairs $(x,u) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$ satisfy problem (1)–(3). We call the function $\tilde{u}$ the optimal control.

Remark 1. A feasible element of problem (1)–(3), (6) is a pair $(x,u) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$ satisfying problem (4), (5) for which $J(x,u) < +\infty$. Since the set $\mathfrak{U}_{ad} \neq \emptyset$, then for any $u \in \mathfrak{U}_{ad} \subset \mathfrak{U}$, by Theorem 1, there exists a unique solution $x = x(u)$ to problem (1)–(3), (6).

Let us formulate a theorem on the existence of optimal control.

Theorem 2. *[4] Let conditions **(S1)**, **(S2)** be satisfied. Then, for any $x_0 \in \mathfrak{A} \cap L^4$, $x_1 \in H^{1,0}$, $T \in \mathbb{R}_+$, there exists a solution to problem (1)–(3), (6).*

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Numerical Methods for Solving Non-convex Optimal Control Problems Based on the Linear Connectivity Property of the Reachable Set

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A curvilinear search algorithm based on the linear connectivity of the reachable set for solving non-convex optimal control problems is proposed. This method allows us to construct control variations that lead to corresponding phase trajectories on the reachable set. Along these trajectories, we solve the auxiliary problem of minimizing the objective function using the nonlocal parabola method. The effectiveness of this algorithm in solving complex control problems has been confirmed.

Keywords: keywords nonlinear optimal control problem, curvilinear search, linear connectivity property

1 Introduction

One approach for solving non-convex optimal control problems problems with a free right end of the trajectory:

$$\dot{x} = f(x(t), u(t), t),\ x(t_0) = x^0,\ u \in R^m : \underline{u}_j \leq u_j \leq \overline{u}_j,\ j = \overline{1, m},$$

$$I(x, u) = \varphi(x(t_1)) \to \min,$$

describing dynamic processes defined at a given time interval $t \in T = [t_0, t_1]$. Such statements arise in the scientific, technical, and industrial fields, and can be reduced to an initial problem of a finite-dimensional optimization problem with a small number of variables in the final phase space. This is determined by at least one function in the reachable set of the controlled dynamical system (see, for example, [1]). So, the problem of approximating the reachable set of a controlled dynamic system is reduced to the following statement: $R_{x_0,t_0} \subset R^n \times R$:

$$R_{x_0,t_0} = \{(x(t_1), t_1) : x(t), t_0 \leq t \leq t_1, \text{is a trajectory starting at}\, (x(t_0), t_0)\}\,.$$

Unfortunately, the problem of approximating the reachable set in a numerical solution is no simpler than the problem of finding a minimum of a non-convex functional, but the properties of this set, known from theoretical studies (see, e.g., [2], [3]), can be used to develop specialized algorithms for optimization of nonlinear controlled dynamical systems. In particular, linear connectivity of the set allows us to create continuous control variations leading to a corresponding trajectory on the set.

2 The main results

The paper proposes a method for scanning the reachable set with simultaneous control improvement. The implemented algorithm relies on a scheme for successive variations in the control space, which uses linear, quadratic and cubic ways of combining record and auxiliary controls with projection onto a reachable set. A control variation (in this case, a quadratic one) is constructed:

$$u^k(\alpha, t) = \alpha^2 \left(\frac{\overline{u}^1(t) + \overline{u}^2(t)}{2} - u_{rec}(t) \right) + \alpha \frac{\overline{u}^2(t) - \overline{u}^1(t)}{2} + u_{rec}(t).$$

Then $u^k(\alpha, t)$ is projected onto the valid region when $t \in T = [t_0, t_1]$: if $u^k(\alpha, t) < \underline{u}$, then $u^k(\alpha, t) = \underline{u}$, if $u^k(\alpha, t) > \overline{u}$, then $u^k(\alpha, t) = \overline{u}$. After that the problem of non-convex optimization is solved: $\min_{\alpha \in [-1,1]} I(u^k(\alpha))$. if $I(u^k(\alpha^*)) < I_{rec}$, then $I_{rec} = I(u^k(\alpha^*))$ and $u_{rec}(t) = u^k(\alpha^*, t)$, here I_{rec} is the best value of the functional at the current iteration of the algorithm and $u_{rec}(t)$ is the control at which it is achieved.

Control variations are projected onto the terminal phase space in the form of curves covering the reachable set (curvilinear search). To construct a one-dimensional search space at each iteration, a set of stochastic auxiliary controls is used, formed applying implemented algorithms for generating controls of a predetermined structure. When solving auxiliary problems of searching for the global extremum of a one-dimensional function, the non-local version of the parabola method is used, allowing one to take into account existing information and perform trials in the local neighborhood of the record control. After completing a specified number of iterations of the considered method, the accuracy of achieving a local extremum is verified using a software implementation of a modification of the reduced gradient method [4].

Curvilinear search algorithms based on the linear connectivity property of the reachable set have been tested using a test collection of optimal control problems, allowing for comparison of the algorithms and an exploration of their limiting properties [5]. In all problems used for testing, the implemented algorithms found the best known solution within an acceptable computational time. Comparison of the obtained numerical simulation results with known reachable sets of the systems is allowed us to conclude that they are sufficiently densely covered by phase trajectories, i.e., the adaptability of the scanning capabilities of the method and its effectiveness in finding optimal controls in optimization problems of nonlinear controlled dynamic systems.

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On the Hurwitz Stability Conditions via Finite Hankel Matrices for Regular Matrix Polynomials

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We develop a Hurwitz stability conditions for regular matrix polynomials via column reduction, generalizing existing approaches constrained by the monic assumption and thus serving as a more natural extension of Gantmacher's classical stability criterion via Markov parameters. The testing framework employs two finite Hankel matrices, whose rectangular blocks are the submatrices of the Markov parameters redefined through a column-wise splitting for column reduced matrix polynomials. We establish an explicit interrelation between the inertias of column reduced matrix polynomials and the derived structured matrices. Furthermore, we demonstrate that the Hurwitz stability of column reduced matrix polynomials can be determined by the Hermitian positive definiteness of these finite Hankel matrices.

Keywords: Hurwitz stability, matrix polynomials, Markov parameters, Hankel matrices, column reduction

A Galerkin Finite Element Method for a Class of Time–Space Fractional Differential Equation with Nonsmooth Data

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In this article, a Galerkin finite element approximation for a class of time-space fractional differential equation is studied, under the assumption that $u_{tt}, u_{2\alpha,tt}, u_{ttt}$ are continuous for $\Omega \times (0, T]$, but discontinuous at time $t = 0$. In spatial direction, the Galerkin finite element method is presented. And in time direction, a Crank-Nicolson time-stepping is used to approximate the fractional differential term, and the product trapezoidal method is employed to treat the temporal fractional integral term. By using the properties of the fractional Ritz projection and the fractional Ritz-Volterra projection, the convergence analyses of semi-discretization scheme and full discretization scheme are derived separately. Due to the lack of smoothness of the exact solution, the numerical accuracy does not achieve second order convergence in time, which is $O(k^{3-\beta}+k^3 t_{n+1}^{-\beta}+k^3 t_{n+1}^{-\beta-1})$, $n = 0, 1, \cdots, N-1$. But the convergence order in time is shown to be greater than one. Numerical examples are also included to demonstrate the effectiveness of the proposed method.

Keywords: time-space fractional differential equation, Riesz fractional derivative, Crank-Nicolson scheme, product trapezoidal method, finite element method, fractional Ritz-Volterra projection

1 The main results

The paper is concerned with Galerkin finite element/Crank-Nicolson method for a class of time-space fractional differential equation

$$u_t - \frac{\partial^{2\alpha} u}{\partial |x|^{2\alpha}} = J_{0,t}^{1-\beta} \frac{\partial^{2\alpha} u}{\partial |x|^{2\alpha}} + f(x,t) \tag{1}$$

subject to the boundary condition $u(0,t) = u(1,t) = 0, t \in J$,
and the initial condition $u(x,0) = u_0(x), x \in \Omega$, where $J = [0,T], \Omega = [a,b]$, and $\frac{\partial^{2\alpha}}{\partial |x|^{2\alpha}}$ represents the Riesz fractional differential operator, defined as $\frac{\partial^{2\alpha}}{\partial |x|^{2\alpha}} = C_\alpha ({}_{RL}D_{a,x}^{2\alpha} + {}_{RL}D_{x,b}^{2\alpha})$, where $C_\alpha = -\frac{1}{2\cos(\pi\alpha)}, 1/2 < \alpha < 1$, which is a linear combination of the left and right Riemann-Liouville derivative with order 2α. It is commonly referred to as anomalous diffusion, where the underlying stochastic process is a Lévy α stable flight. $J_{0,t}^{1-\beta} (0 < \beta < 1)$ is the temporal fractional integral operator with singular kernel $t^{-\beta}/\Gamma(1-\beta)$.

Theorem 1 Let $u(t)$ and $u_h(t)$ be the exact solutions and semi-discrete solution respectively, then we have the following error estimate

$$\|u_h(t) - u(t)\| \le Ch^r(\|u_0\|_r + \int_0^t \|u_t(s)\|_r ds). \tag{2}$$

Theorem 2 Assume that $u(t_{n+1})$ and U^{n+1} are the exact solutions and fully discrete solution at time $t = t_{n+1}$ respectively, then

$$\max_{0 \le n \le N-1} \|U^{n+1} - u(t_{n+1})\| = O(h^r + k^{3-\beta} + k^3 t_{n+1}^{-\beta} + k^3 t_{n+1}^{-\beta-1}). \tag{3}$$

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A Theorem on the Parameter Dependence of Coincidence Points and Applications to the Study of a Class of Functional Differential Equations*

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A theorem on the existence and continuous parameter dependence of coincidence points of two mappings of metric spaces is formulated. One of the mappings is assumed to be regular, and the second is the limit of a sequence in a set of compact mappings. This result applies to the study of a Volterra functional differential equation that is not solved for the derivative of the unknown function.

Keywords: coincidence point of mappings, regular mapping, functional differential equation, Tonelli's method

Let (X, ρ_X), (Y, ρ_Y) be metric spaces. Let mappings $F, G : X \to Y$ be given. The set of their coincidence points will be denoted by $\mathrm{Coin}(F, G)$, i.e. $\mathrm{Coin}(F, G) := \{x \in X \mid F(x) = G(x)\}$. The existence and estimates of coincidence points for a covering mapping and a Lipschitz mapping were studied by A.V. Arutyunov in [1]. The theorem proved in that work is a natural extension of the Banach fixed point theorem to coincidence points. For coincidence points, an analogue of another classical fixed point principle, namely the Schauder theorem, has not yet been obtained in the literature. The present research aims to contribute to solving this problem.

A mapping $F : X \to Y$ will be called generalized regular if the following condition holds:

$$\forall y \in F(X) \ \ \forall \varepsilon > 0 \ \ \exists \delta(y, \varepsilon) > 0 \ \ \forall \tilde{y} \in F(X)$$
$$\rho_Y(y, \tilde{y}) < \delta \ \Rightarrow \ \forall x \in F^{-1}(y) \ \ \exists \tilde{x} \in F^{-1}(\tilde{y}) \ \ \rho_X(x, \tilde{x}) < \varepsilon.$$

Obviously, a covering mapping (also called metrically regular) possesses the property of being generalized regular.

Let mappings $G_i : X \to Y$, $i \in \mathbb{N}$, be given.These mappings are called collectively compact on a set $U \subset X$ if the set $\bigcup_{i \in \mathbb{N}} G_i(U)$ is relatively compact in Y. A sequence $\{G_i\}$ is said to converge continuously to a mapping $G : X \to Y$ on a set $U \subset X$, if for any $x, x_i \in U$, $i \in \mathbb{N}$, such that $\rho_X(x_i, x) \to 0$, we have $\rho_Y(G_i(x_i), G(x)) \to 0$.

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Theorem 1. *Let the metric space X be complete; let the mappings $G_i : X \to Y$, $i \in \mathbb{N}$, be collectively compact on a closed set $U \subset X$, and let their sequence $\{G_i\}$ converge continuously to a mapping $G : X \to Y$ on the same set U; let the mapping $F : X \to Y$ be continuous and generalized regular; for any $y \in F(U)$ let the set $F^{-1}(y) \subset X$ be compact, and let $U \bigcap \mathrm{Coin}(F, G_i) \neq \emptyset$ for all $i \in \mathbb{N}$. Then any sequence $\{x_i\}$, with $x_i \in U \bigcap \mathrm{Coin}(F, G_i)$ is relatively compact in X and all its limit points belong to the set $\mathrm{Coin}(F, G)$.*

Theorem 1 can be used to study some functional differential equations that are not solved for the derivative of the unknown function. In particular, it is shown below that Theorem 1 makes it possible to extend the Tonelli's method, known as a method for studying and approximately solving equations solved for the derivative, to such equations.

In the standard way, we denote by $L^n_{\infty[0,T]}$, $C^n_{[0,T]}$ and $AC^n_{[0,T]}$ the Banach spaces of essentially bounded measurable, continuous, and, respectively, absolutely continuous functions $[0,T] \to \mathbb{R}^n$. Let functions $f, g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$, satisfying the Caratheodory conditions, a continuous Volterra mapping $H : C^n_{[0,T]} \to L^n_{\infty[0,T]}$ and vector $x_0 \in \mathbb{R}^n$ be given. Consider the Cauchy problem

$$f(t, \dot{x}(t)) = g(t, (Hx)(t)), \quad t \in [0,T]; \quad x(0) = x_0. \tag{1}$$

By a solution we mean an absolutely continuous function on some interval $[0,\eta] \subset [0,T]$ that satisfies the equation for almost every $t \in [0,\eta]$ and the initial condition.

Let $\tau \in (0,T]$. For any $x \in AC^n_{[0,T]}$ we denote $(S_\tau x)(t) = \begin{cases} x(t-\tau), & \text{if } t \in (\tau, T], \\ x(0), & \text{if } t \in [0,\tau]. \end{cases}$

Let $\tau_i > 0$, $i \in \mathbb{N}$, be such that $\tau_i \to 0$. Consider the "approximate" Cauchy problem

$$f(t, \dot{x}(t)) = g(t, (HS_{\tau_i} x)(t)), \quad t \in [0,T]; \quad x(0) = x_0. \tag{2}$$

Suppose there exist $R > 0$, $r > 0$, $\Delta \in (0,T]$ such that for almost every $t \in [0,\Delta]$

$$\varphi(t) := \sup_{u \in \mathrm{B}_R} |f(t,u)|_{\mathbb{R}^n} < \infty, \quad f(t, \mathrm{B}_R) \supset g(t, \mathrm{D}_r(t)),$$

where

$$\mathrm{B}_R := \{x \in \mathbb{R}^n : |x|_{\mathbb{R}^n} \leq R\}, \quad \mathrm{D}_r(t) := \{(Hu)(t) : u \in C^n_{[0,\Delta]},\ \forall s \in [0,\Delta]\ |u(s) - x_0|_{\mathbb{R}^n} \leq r\}.$$

Suppose further that for almost every $t \in [0,\Delta]$ the function $\phi_t(\cdot) := f(t, \cdot)$ satisfies

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \forall u, \tilde{u} \in \phi_t(\mathrm{B}_R)$$
$$|u - \tilde{u}|_{\mathbb{R}^m} < \delta \varphi(t) \ \Rightarrow \ \forall x \in \mathrm{B}_R \cap \phi_t^{-1}(u) \ \exists \tilde{x} \in \mathrm{B}_R \cap \phi_t^{-1}(\tilde{u}) \ \ |x - \tilde{x}|_{\mathbb{R}^n} < \varepsilon.$$

Under these conditions, there exists $\eta > 0$ such that for each $i \in \mathbb{N}$ problem (2) has a solution $x = x_i(t)$, $t \in [0,\eta]$, whose derivative satisfies almost everywhere on $[0,\eta]$ the inclusion $\dot{x}_i(t) \in \mathrm{B}_R$. Denote by $\mathrm{Sol}_i(\eta, \mathrm{B}_R)$ the set of all such solutions of problem (2). From Theorem 1 it follows that any sequence $\{\dot{x}_i\}$, where $x_i \in \mathrm{Sol}_i(\eta, \mathrm{B}_R)$, is relatively compact in $L_{\infty[0,\eta]}$ and every its limit point is the derivative of some solution of problem (1).

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On the Fixed Points of the Mapping $[ax + b(x)]$

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This paper investigates the fixed points of the function $f(x) = [ax+b(x)]$, where $a \in \mathbb{R}$, $b(x)$ is a real-valued function, and $[\cdot]$ denotes the floor (greatest integer) function. Due to the discontinuous nature of the floor operator, the dynamical behavior of such functions differs significantly from that of continuous linear maps. We establish necessary and sufficient conditions for the existence of fixed points, that is, points x satisfying $f(x) = x$. The analysis is carried out under various assumptions on the parameter a (including the cases $a < 1$, $a = 1$, and $a > 1$) and on the structure of the perturbation term $b(x)$. Explicit characterizations of the fixed point set are obtained in terms of inequalities derived from the definition of the floor function.

Keywords: dynamical systems, floor function, fixed point

By Rozikov et al. [1], who studied the one-parameter family of discretised one-dimensional linear maps given by $f : \mathbb{R} \to \mathbb{R}, f(x) = [\lambda x]$, where $\lambda \in \mathbb{R}$. For every $\lambda \in \mathbb{R}$ authors studied corresponding dynamical systems and described of all fixed points of f. In [2], the author generalized the dynamical system for the case $f : \mathbb{R} \to \mathbb{R}, f(x) = [\lambda x + b]$, where $\lambda, b \in \mathbb{R}$ and studied this dynamical systems. In our case generalized both cases, i.e. we study the dynamical system associated with the function $f : R \to R$ defined by

$$f(x) = [ax + b(x)], \tag{1}$$

where $a \in \mathbb{R}\ 0$ is a parameter and $b(x)$ is a function. $E(b)$ is the set of values of the function $b(x)$.

Definition. A point $x \in \mathbb{R}$ is called fixed point of f if $f(x) = x$. The set of all fixed points is denoted by $Fix(f)$.

The dynamics of integer-valued transformations generated by the floor function have attracted considerable attention in recent years. In particular, the fixed points of the function

$$f(x) = [ax],$$

corresponding to the case $b(x) = 0$, for all $x \in \mathbb{R}$ (see, e.g., [1]), have been thoroughly studied. This mapping exhibits nontrivial behavior due to the discontinuity of the floor function. The case of a constant perturbation, namely $b(x) = b$, for all $x \in \mathbb{R}$ (see, e.g., [2]), has also been investigated, revealing additional structural properties of the resulting dynamical system.

The following theorem gives all fixed points of function (1).

Theorem. For the set of fixed points of function (1), the followings hold

I. Let $a = 1$. Then

I.1. For any $E(b) = [0, 1)$,

$$Fix(f) = \mathbb{Z}.$$

I.2. For any $E(b) \subset \mathbb{R}\ [0, 1)$,

$$Fix(f) = \emptyset.$$

II. Let $a \neq 1$.

II.1. If $a < 1$, then

$$Fix(f) = \left\{ x \in \mathbb{Z} : \ \frac{1}{a-1} < x + \frac{b(x)}{a-1} \leq 0 \right\}.$$

II.2. If $a > 1$, then

$$Fix(f) = \left\{ x \in \mathbb{Z} : \ 0 \leq x + \frac{b(x)}{a-1} < 1 \right\}.$$

Remark. Note that the theorem coincides with Theorem 1 of [2] in the case $b(x) \equiv b$ for all $x \in \mathbb{R}$.

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Problem of Data Redundancy Minimization under Information Completeness Constraints: Algorithm and Efficiency Estimate

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The paper addresses the problem of reducing the volume of historical data in the 1C: Payroll and Human Resource Management 3.1 (1C: ZUP 3.1) configuration, which leads to performance degradation. Standard 1C tools lack an automated mechanism for data convolution that complies with Russian legislation. An external processing tool is developed to automate the aggregation of current balances (personnel, leave, salaries) as of a specified date and to mark outdated documents for deletion. The algorithm ensures recoverability from failures and is tested on both demo and production databases, demonstrating a reduction in database size by up to 20% and a significant increase in operational speed.

Keywords: data redundancy, data convolution, payroll accounting, personnel records, historical data aggregation

1 Introduction

In modern enterprises, the 1C: ZUP 3.1 configuration accumulates a vast amount of historical data over time. This leads to several critical issues: slower document processing, difficulties in creating backups, and prolonged system update times [1]. While mechanisms exist to transfer summary data to an accounting system (e.g., 1C: Accounting), there are no standard tools in 1C: ZUP 3.1 to safely compress historical data (e.g., old payroll calculations, outdated timesheets) while retaining legally required records.

This paper presents an external processing tool for data convolution. The primary goal is to automate the import stage when integrating a new organization into an existing accounting system by preparing aggregated data and cleaning up redundant information.

2 Problem statement and data analysis

The 1C: Enterprise 8.3 platform stores all operational changes as documents generating movements in accumulation and information registers. The primary sources of database bloat are:

- ❑ accumulation registers (accumulate all movements of funds and accruals);
- ❑ documents: “Payroll”, “Statements for salary payments”, “Vacation”, “Sick leave”;
- ❑ personnel documents: “Hiring”, “Transfer”, “Dismissal”.

Russian legislation (Federal Law No. 402-FZ, Order of the Rosarchive No. 236) establishes different retention periods [2,3]. Long-term storage applies to personal income tax certificates and personal accounts, which must be excluded from deletion. Short-term storage applies to payroll documents, statements, and vacation orders, which are subject to aggregation and removal.

Thus, the task is to preserve critical data with long retention periods and aggregate short-term data into initial balance documents.

3 Designed algorithm

The developed convolution process consists of five sequential stages:

1. Preparation: creating a backup, blocking user work.
2. Analysis and extraction: reading data from registers as of the convolution date (finding current employees, balances, leave balances).
3. Aggregation (creation of opening balance documents):
   - ❑ “Initial staffing arrangement” replaces the history of personnel orders.
   - ❑ “Initial salary arrears” accumulates current debts to employees.
   - ❑ “Data transfer” stores information for calculating average earnings.
4. Cleaning (marking for deletion): all historical documents prior to the convolution date are marked for deletion, except for long-term storage documents.
5. Completion: unblocking the database, generating a report.

The algorithm includes a checkpoint mechanism to ensure recoverability in case of failures. The status of each stage is saved in a service document; upon restart, the process continues from the last completed stage.

4 Testing results

Testing was performed on a demo database (189.25 MB, 31 employees) and a production database (4.25 GB, over 450 employees).

Key verification criteria: - comparison of the "Staffing arrangement" report before and after convolution showed complete identity; - the "Payroll and contributions" document for the period after convolution yielded the same amounts as in the original database; - the calculation of vacation pay based on transferred average earnings data was error-free.

Performance results (production database):

- total execution time: 6–7 hours;
- number of documents marked for deletion: 17,000;
- time for marking (batch processing): 4.5 hours;
- database size reduction: 20% (from 4.25 GB to 3.4 GB).

The reduction in the demo database was about 48% (from 189.25 MB to 98.42 MB).

5 Conclusion

The developed external processing tool automates the data convolution process for the 1C: ZUP 3.1 configuration, solving the problem of performance degradation due to historical data accumulation. The algorithm accounts for Russian legal requirements regarding data retention, ensures recoverability from failures, and provides full control over the process through a user-friendly interface. Testing confirmed the functional correctness (preservation of current data) and the performance efficiency, which accelerates system operations by more than 50% compared to manual methods.

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Impact of Gradient Estimation Accuracy on the Convergence of Second-Order Algorithms in Nonlinear Analysis Problems

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This paper investigates the robustness of Newton's method and BFGS to floating-point representation errors and gradient estimation noise. It is shown that in nonlinear extremal problems, the accuracy of derivative computation is a critical factor determining the transition from quadratic convergence to divergence.

Keywords: nonlinear optimization, Newton's method, BFGS, computational accuracy, gradient descent

In classical nonlinear analysis, convergence proofs for second-order algorithms typically rely on the assumption that derivatives are computed with sufficient accuracy. However, when implemented in floating-point arithmetic (IEEE 754), unavoidable computational errors arise. It is known that errors in the descent direction can degrade quadratic convergence to linear or cause complete divergence of the process near narrow extrema [3].

We consider the unconstrained minimization problem $\min_{x\in\mathbb{R}^n} f(x)$, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function. The goal of this work is to quantitatively assess the influence of gradient computation errors on iterative stability. We study iterative processes of the form:

$$x_{k+1} = x_k - \alpha_k B_k^{-1} \nabla f(x_k), \tag{1}$$

where B_k is an approximation of the Hessian $\nabla^2 f(x_k)$: exact in Newton's method and quasi-Newtonian in BFGS. The primary focus is on the analysis of the condition number $\operatorname{cond}(B_k)$.

A software suite was developed in Python (NumPy library) that allows gradient accuracy to be varied by introducing controlled noise and by changing numerical data types: `float16`, `float32`, `float64`. Experiments were conducted on classical multiextremal benchmark functions: the Rosenbrock and Rastrigin functions.

The computational experiments revealed a *plateau* effect — a situation in which the algorithm ceases to converge because the gradient noise becomes comparable in magnitude to the true gradient value.

It was found that under noisy gradient conditions, the matrix B_k in BFGS rapidly loses positive definiteness. To compensate for this effect, an adaptive Tikhonov regularization was implemented:

$$\tilde{B}_k = B_k + \lambda I, \quad \lambda = \eta \cdot \max(\operatorname{diag}(B_k)), \tag{2}$$

where η is a parameter depending on the numerical precision used.

It is shown that there exists a threshold accuracy level below which second-order methods lose their advantage over gradient descent, due to the high cost of Hessian recomputation when the resulting descent direction is of poor quality.

The aim of this work is to bridge the theoretical foundations of nonlinear analysis with applied problems in numerical optimization. The results obtained provide practical recommendations for the choice of numerical precision and optimization methods in machine learning and automated control tasks operating under noisy input data conditions.

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Optimal Control of a Firm's Investment and Consumption Policy

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This paper studies an optimal control problem for a firm that allocates its profit between investment in production and consumption while using a loan. The goal is to maximize the total discounted consumption. The dynamics are bilinear, and the control set is a convex polygon. The Pontryagin maximum principle reduces the maximization of the Hamiltonian to a linear programming problem at each time. The optimal control is constructed explicitly by solving the adjoint Cauchy problem.

Keywords: optimal control, maximum principle, bilinear control system, financial policy

We consider a firm that uses its profit and an attracted loan for development. The firm allocates part of its profit to investment and the remainder to consumption. The goal is to maximize the total discounted consumption. The mathematical model is given by (1):

$$
\begin{aligned}
&\Phi(u) = \int_0^T (1 - u_1(t) - u_2(t))x(t)e^{-pt}dt \to \max, \\
&\dot{x} = (cu_1 + u_2)rx, \quad x(0) = x_0, \\
&U = \left\{u \in \mathbb{R}^2; u_1 \geq 0, 0 \leq u_2 \leq 1, cu_1 + u_2 \leq \frac{g}{r}\right\}.
\end{aligned} \tag{1}
$$

Here $x(t)$ is the firm's capital, u_1 and u_2 are the shares of profit allocated to investment and loan repayment respectively, $r > 0$ is investment efficiency, $0 < c < 1$ is loan interest rate, $cu_1 + u_2 \leq g/r$ is debt service constraint, p is discount rate, T is a planning horizon. The objective is to maximize the total discounted consumption, where e^{-pt} is the discount factor.

The Pontryagin maximum principle is applied. The Hamiltonian is $H = \psi(cu_1 + u_2)rx + (1 - u_1 - u_2)xe^{-pt}$. Since $x > 0$, the maximization of H w.r.t. u is equivalent to a linear programming problem in (u_1, u_2). Solving it graphically yields a piecewise constant control structure. The adjoint variable $\psi(t)$ satisfies the following differential equation:

$$
\dot{\psi} = -\frac{\partial H}{\partial x} = -((cu_1 + u_2)r\psi + (1 - u_1 - u_2)e^{-pt}),
$$

with transversality condition $\psi(T) = 0$. The optimal control is constructed by solving this Cauchy problem backward in time. The sufficient optimality condition [1] guarantees that the obtained control is indeed optimal.

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Description of $G_2^{(2)}$-periodic Ground States for the $(m+1)$-state Chui-Weeks Model

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In this work, we undertake a rigorous examination of the $(m+1)$-state Chui-Weeks model. We describe all $G_2^{(2)}$-periodic ground states associated with this model on a second-order Cayley tree.

Keywords: Cayley tree, Chui-Weeks model, ground state, translation-invariant ground state, $G_2^{(2)}$-periodic ground state

The Cayley tree Γ^k (see, e.g., [2,3]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called ***nearest neighboring vertices***, and we write $l = \langle x, y \rangle$.

It is known (see [2]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e, a_i\}$, $i = 1, ..., k+1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, ..., a_{k+1}$.

We consider a model in which the spin variable takes values in the set $\Phi = \{0, 1, 2, ..., m\}$. For $A \subseteq V$ a spin ***configuration*** σ_A on A is defined as a function $x \in A \mapsto \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega_A = \Phi^A$. Denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

Let $G_k/G_k^* = \{K_0, K_1 ..., K_{r-1}\}$ be a quotient group, where G_k^* is a normal subgroup of index $r \geq 1$.

Definition 1. *A configuration $\sigma \in \Omega$ is called G_k^* -periodic, if $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^* \subset G_k$.*

For a given periodic configuration the index of the subgroup is called the ***period of the configuration***.

Definition 2. *A configuration that is invariant with respect to all shifts is called translation-invariant.*

The Chui-Weeks model (see [1,4,5]) is defined by the following Hamiltonian

$$H(\sigma) = J \sum_{\langle x,y \rangle \in L} \mid \sigma(x) - \sigma(y) \mid + \alpha \sum_{x \in V} \delta_{\sigma(x),0}, \tag{1}$$

where $J, \alpha \in \mathbb{R}$, α is an external field, and $\sigma \in \Omega$ as well as $\delta_{i,j}$ denotes the Kronecker delta, that is, $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

We study all $G_2^{(2)}$-periodic ground states for the Chui-Weeks model on the Cayley tree of order two, where

$$G_2^{(2)} = \{x \in G_2 : \mid x \mid \ \text{ is even}\},$$

where $\mid x \mid$ means length of the word x.

All $G_2^{(2)}$-periodic configurations have the following form:

$$\sigma(x) = \begin{cases} \sigma_0, \text{ if } x \in G_2^{(2)}, \\ \sigma_1, \text{ if } x \in G_2 \;\; G_2^{(2)}, \end{cases}$$

where $\sigma_0, \sigma_1 \in \Phi$.

Recall that if a $G_2^{(2)}$-periodic configuration is a ground state, then we call it a $G_2^{(2)}$-periodic ground state.

The following theorem describes all $G_2^{(2)}$-periodic ground states for the $(m+1)$-state Chui-Weeks model.

Theorem 1. *For the Chui-Weeks model, the following assertions hold:*
(i) The $G_2^{(2)}$-periodic configurations

$$\sigma(x) = \begin{cases} i, \text{ if } x \in G_2^{(2)}, \\ j, \text{ if } x \in G_2 \;\; G_2^{(2)}, \; i, j \in \Phi \;\; \{0, m\}, i \neq j, \end{cases}$$

are $G_2^{(2)}$-periodic ground states iff $J = 0, \alpha \geq 0$;
(ii) The $G_2^{(2)}$-periodic configurations

$$\sigma(x) = \begin{cases} 0, \text{ if } x \in G_2^{(2)}, \\ m, \text{ if } x \in G_2 \;\; G_2^{(2)}, \end{cases} \quad \sigma(x) = \begin{cases} m, \text{ if } x \in G_2^{(2)}, \\ 0, \text{ if } x \in G_2 \;\; G_2^{(2)} \end{cases}$$

are $G_2^{(2)}$-periodic ground states iff $J \leq 0, \alpha = 0$;
(iii) Moreover, there exist no other non-translation-invariant $G_2^{(2)}$-periodic ground states besides those described in parts (i) and (ii).

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Discrete Dynamical Systems for Consistency Restoration in Multi-Agent Systems: A Categorical Approach via Asymmetric Learning Lenses

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Modern AI agents combine reasoning, tool use, code generation, validation, and iterative correction. A categorical interpretation of such systems is proposed based on asymmetric learning lenses with amendment. Agent execution is treated as a parameter-dependent transformation, while validation feedback is represented as a delta inducing parameter adaptation, source-state repair, and target-side amendment.
For multi-agent systems, an adaptive protocol component is introduced, so that repair may affect not only local agent parameters but also the interaction structure. The resulting repair loop is modeled as a discrete dynamical system on the space of states, parameters, and protocols. This provides a compositional language for studying consistency restoration, feedback propagation, and convergence of agent-based computational systems.

Keywords: asymmetric learning lenses, bidirectional transformations, AI agents, multi-agent systems, repair dynamics

1 The main results

Delta lenses and their learning extensions provide a compositional language for change propagation and supervised repair [1,2]. This point of view is adapted to AI-agent architectures in which validation errors trigger iterative modification of prompts, generated code, memory states, tool configurations, and communication protocols.

Let S be a category of source states, V a category of observable outputs, and P a category of parameters. Agent execution is described by a functor

$$\mathrm{get} : P \times S \to V.$$

For objects $p \in \mathrm{Ob}(P)$ and $s \in \mathrm{Ob}(S)$, suppose that validation feedback is represented by a delta

$$v : \mathrm{get}(p, s) \to y^{\Delta}.$$

The repair mechanism computes

$$\begin{aligned} e &= \mathrm{put}^{\mathrm{upd}}_{p,s}(v) : p \to p', \\ u &= \mathrm{put}^{\mathrm{req}}_{p,s}(v) : s \to s', \\ a &= \mathrm{put}^{\mathrm{self}}_{p,s}(v) : y^{\Delta} \to \mathrm{get}(p', s'). \end{aligned}$$

Here e is parameter adaptation, u is source-state repair, and a is target-side amendment. Since $(e, u) : (p, s) \to (p', s')$ is a morphism in $P \times S$, the functor get induces a morphism

$$\mathrm{get}(e, u) : \mathrm{get}(p, s) \to \mathrm{get}(p', s').$$

The repaired consistency condition is

$$\mathrm{get}(e, u) = a \circ v.$$

Thus, the repaired parameter and source state produce the same observable change as the feedback delta followed by the amendment.

An *Agent Adaptive Lens* is the tuple

$$\mathcal{A} = (S, V, P, \mathrm{get}, \mathrm{put}^{\mathrm{upd}}, \mathrm{put}^{\mathrm{req}}, \mathrm{put}^{\mathrm{self}})$$

satisfying the repaired consistency condition above.

For a multi-agent system with n agents, let G, M, Γ, S_i, and P_i be categories of global task states, shared memories, protocols, local agent states, and local agent parameters, respectively. Define

$$S_{\mathrm{MAS}} = G \times M \times \prod_{i=1}^{n} S_i, \qquad P_{\mathrm{MAS}} = \Gamma \times \prod_{i=1}^{n} P_i.$$

At the schematic level, the corresponding multi-agent adaptive lens has the compositional form

$$L_{\mathrm{MAS}} = L_{\mathrm{agg}} \circ \left(\bigotimes_{i=1}^{n} L_i \right) \circ L_\Gamma,$$

where L_Γ interprets the protocol, $\bigotimes_i L_i$ denotes parallel composition of local agent lenses, and L_{agg} aggregates their outputs. Hence feedback may induce both local updates

$$p_i \to p_i', \qquad s_i \to s_i',$$

and protocol updates

$$\gamma \to \gamma', \qquad \gamma, \gamma' \in \mathrm{Ob}(\Gamma).$$

The repair dynamics is considered on the set

$$X = \mathrm{Ob}(S_{\mathrm{MAS}} \times P_{\mathrm{MAS}}).$$

A repair loop is modeled as a discrete dynamical system

$$x_{t+1} = F_{\delta_t}(x_t), \qquad F_{\delta_t} : X \to X,$$

where δ_t is the validation feedback at step t. To connect this model with optimization and nonlinear discrete dynamics, an inconsistency functional is introduced:

$$E : X \to \mathbb{R}_{\geq 0}.$$

The condition $E(x) = 0$ means that the current state is consistent with the validation criterion. A repair strategy is error-reducing if

$$E(F_\delta(x)) < E(x)$$

whenever $E(x) > 0$. If the set $E(X)$ has no infinite strictly decreasing chains, in particular if it is finite, then such a strategy reaches a consistent state in finitely many repair steps. If monotonicity fails, the same dynamical representation allows one to study cycles, local traps, and protocol-level failure modes.

The proposed framework gives a categorical and dynamical abstraction for repair loops in single-agent and multi-agent computation. Its main contribution is a compositional language for consistency restoration, adaptive validation, and protocol repair.

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Problem of Classification and Geolocation of Construction Images Based on Convolutional Neural Networks and Spherical Trigonometry

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A prototype of a web application has been developed for automatic classification of construction site photographs (facade/roof) and their binding to the address of an apartment building. The system uses a MobileNetV2-based convolutional neural network trained via transfer learning on a small dataset (1300 images) and a geolocation module that determines the building address from GPS coordinates and azimuth extracted from EXIF metadata using spherical trigonometry and OpenStreetMap data. The backend is implemented on the asynchronous FastAPI framework with PostgreSQL and LDAP authentication; secure remote access is provided through OpenVPN and UserGate NGFW. The classifier achieved 96.88% validation accuracy, and the geolocation module correctly identified the address in 94% of cases. The economic effect consists in reducing labor costs by 95% (about 15,000 man-hours per year) with a payback period of less than one month.

Keywords: image classification, convolutional neural networks, geolocation, spherical trigonometry

1 Introduction

In the field of capital repair of apartment buildings, daily photo-fixation of work stages is carried out: facades, roofs, utilities, etc. Employees make dozens of pictures, which are then manually sorted by addresses and types of work, stored in network folders. With a staff of 40 specialists processing 100–200 images per day, annual labor costs for routine classification reach 8,300–16,700 man-hours. Manual processing leads to errors: photos end up in wrong folders, defect types are incorrectly determined, and critical data is lost. The aim of this work is to create a software and hardware complex that automatically classifies images by type (facade/roof) and binds them to the address of the building using multimodal data (visual content and GPS/azimuth metadata).

2 Methods and architecture

2.1 Neural network classifier

The classification function $f_{\text{class}} : \mathcal{I} \to \mathcal{C}$ distinguishes two classes: "facade" and "roof". The training dataset consisted of 1300 images (650 per class), split into training (80%) and validation (20%) sets. The baseline model (Base CNN) with three convolutional layers (16, 32, 64 filters) and max-pooling, followed by a dense layer and softmax output, showed strong overfitting: training accuracy 98.8%, validation accuracy only 66.9%.

To overcome overfitting, we applied transfer learning with MobileNetV2 pre-trained on ImageNet, replacing the top with GlobalAveragePooling2D, a dense layer (128 neurons), batch normalization, dropout (0.3), and a two-class softmax output. Data augmentation (rotation, shift, zoom, horizontal flip) and regularization techniques allowed reaching validation accuracy of 96.88% with a loss of 0.12, while the gap between training and validation metrics dropped to less than 2%.

2.2 Geolocation module

The function $f_{\text{geo}} : \mathcal{I} \to \mathcal{A}$ determines the address a_j of the building captured in the photo. Two modules work sequentially: `GeolocationService` extracts GPS coordinates (ϕ, λ) and azimuth α from EXIF tags (GPSLatitude, GPSLongitude, GPSImgDirection) using the Pillow library. Coordinates are converted from DMS to decimal degrees.

`BuildingLocator` constructs a sighting ray of length 100 m from the shooting point in the azimuth direction. The Earth is approximated by a sphere of radius $R = 6\,371\,370$ m, and the ray is computed by solving the direct geodetic problem on the sphere via the cosine theorem for a spherical triangle:

$$\phi_1 = \arcsin(\sin\phi_0 \cos\delta + \cos\phi_0 \sin\delta \cos\alpha),$$

$$\lambda_1 = \lambda_0 + \operatorname{arctan2}(\sin\alpha \sin\delta \cos\phi_0,\ \cos\delta - \sin\phi_0 \sin\phi_1),$$

where $\delta = d/R$ is the angular distance. The ray is represented as a LineString (Shapely library). Building polygons within a 100 m radius are loaded from OpenStreetMap via OSMnx or Overpass API, and the nearest intersecting building is selected. The address is extracted from OSM tags or obtained through reverse geocoding with Nominatim. Accuracy evaluated on 100 test photos: correct address identification in 94% of cases; mean spatial error for correctly identified buildings is 8.7 m.

2.3 Web application and security

The server side is built with the asynchronous FastAPI framework (ASGI), PostgreSQL database with asyncpg driver, and LDAP authentication (ldap3) integrated with corporate Active Directory. The frontend is a single-page application (HTML5, CSS3, vanilla JavaScript) communicating via Fetch API. It supports multiple download images, displays classification and geolocation results, and provides personal statistics.

Secure remote access is implemented by a combined scheme: staff connect via OpenVPN tunnel on a MikroTik router (certificates, SSL/TLS), receiving an internal IP and direct access to the server; external users access via UserGate NGFW reverse proxy (HTTPS, port 443), which terminates TLS and forwards traffic to the internal server. User authentication in the application always uses a single Active Directory account.

3 Economic efficiency

Manual classification takes on average 30 seconds per image. With a daily norm of 150 photos per employee and 40 specialists, annual labor costs are approximately 12,338 hours, which at a rate of 800 rubles/hour gives about 9.87 million rubles per year.

4 Conclusion

The prototype of the automatic classification and geolocation system for construction images has been successfully developed and tested. The classifier based on MobileNetV2 achieves 96.88% validation accuracy, the geolocation module correctly determines the address in 94% of cases. The web interface provides convenient batch processing, and the access security scheme meets corporate requirements. The system is ready for pilot operation and can be recommended for implementation in organizations controlling the repair of apartment buildings. Future development includes expanding the set of recognizable classes, creating a mobile application (TensorFlow Lite), and integrating with BIM systems.

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Problem of Technical Specification Generation with Full Context Transfer: Comparison of Stepwise and Monolithic Strategies

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A prototype of a web application has been developed for the automatic generation of technical specifications for a software product based on a text description of a user's idea using large language models (LLM). The promotion system is based on a zero-shot role-playing approach.

Keywords: terms of reference, large language models (LLM), step-by-step generation, prompt-engineering

1 The main results

In the developed web application, it has been experimentally proven that step-by-step technical task generation with the transfer of the full context of previously created sections in each LLM request provides fundamentally higher document quality compared to the monolithic approach: with identical input data, the step-by-step strategy gave an average section length of about 8-9 thousand characters, complete details of functional and non-functional requirements, and also, the absence of logical contradictions and "hallucinations"; whereas the monolithic generation of the same TK led to superficial formulations and the unsuitability of the document for real development, which confirms the key hypothesis about the need to decompose the complex task of generating a multi-part structured document.

A Formal Model of a Turn-Based Tactical RPG System on a Discrete Grid

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This work presents a formal specification of a commercial-scale turn-based isometric strategy game with embedded role-playing game mechanics, implemented in the Unity framework using C#. The game state is defined as a structured tuple comprising a finite set of units, a discrete tactical grid, a multidimensional parameter space for each unit, a finite set of abilities, and a collection of status effects. Key unit parameters include Strength, Magic, Stamina, Initiative, Health, Defense, elemental defenses, and resistance coefficients. All parameters are instantiated via a data-driven architecture, ensuring separation between data schema and behavioral logic. The tactical combat subsystem operates on a grid-based field with stamina-constrained movement, ability execution, and turn scheduling. Enemy artificial intelligence is implemented as a Behavior Tree formalism with a scoring mechanism for target selection based on distance, health percentage, defense values, and elemental resistances. The visual layer is defined as a two-dimensional pixel art style with hand-drawn and animated sprites, while audio is produced using digital audio workstation software. All core mechanical subsystems are fully implemented and tested. The project remains under active development toward commercial release.

Keywords: game development, Unity, turn-based strategy, RPG system, Behavior Tree AI, pixel art

1 The main results

A complete market analysis has been conducted. The project targets the global PC gaming market with strategic attention to the Chinese segment.

The core rule system is fully implemented in Unity using C# [1]. The parameter space includes: Strength (increasing physical damage), Magic (increasing status effect damage), Stamina (action economy), Initiative (turn order), Health, Defense, Fire Defense, Stun Resistance, Fire Resistance, and other parameters. All values are data-driven via ScriptableObjects [2].

The tactical combat system is complete. It operates on a grid-based field where each unit occupies one cell. Movement, ability usage, and turn ending consume Stamina. Damage calculation includes defense multipliers and resistance checks.

Enemy AI is implemented using custom Behavior Trees. The AI evaluates all available targets, calculates scores based on distance, health percentage, defense values, and elemental resistances, then moves intelligently and selects optimal abilities.

The artistic identity is fully defined as a two-dimensional pixel art style with hand-drawn and animated sprites created in Aseprite. Sound design and music are produced in FL Studio. This style is production-efficient while maintaining high artistic cohesion.

The following systems are complete and tested: turn management, grid navigation, ability execution with cooldowns, status effects (burn and stun), inventory with consumable items, and full AI behavior.

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Pareto Front Construction in a Two-Criteria Optimal Control Problem

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A two-criteria optimal control problem is considered for a linear system with scalar control and multidimensional state. The conflicting criteria penalize the state deviation and the control effort, respectively. Admissible controls are piecewise-constant functions. By linear scalarization the problem is reduced to a classical linear-quadratic optimal control problem.

The original infinite-dimensional problem is approximated by a corresponding finite-dimensional quadratic optimization problem on a uniform time grid. A Python class `LQPSolver` implements the complete pipeline: input validation, precomputation of all matrices, optional criterion normalization, and solution for arbitrary weights.

The method is illustrated on the linearized vertical-plane dynamics of a Boeing 737-800 aircraft subject to turbulence. Numerical results demonstrate the trade-off between state recovery speed and control effort and confirm that only convex parts of the Pareto front can be recovered by the scalarization approach.

Keywords: multi-objective optimal control, linear-quadratic regulator, Pareto front, piecewise-constant approximation, Python implementation

Consider the two-criteria optimal control problem

$$J_1(x,u) = \frac{1}{2}\langle x(T), Px(T)\rangle + \frac{1}{2}\int_0^T \langle x(t), Q(t)x(t)\rangle\, dt \to \min,$$

$$J_2(x,u) = \frac{1}{2}\int_0^T u(t)^2\, dt \to \min,$$

subject to the linear dynamics $\dot{x} = A(t)x + b(t)u$, $x(0) = x^0$, where $u \in \mathbb{R}$ is the scalar control, $x \in \mathbb{R}^n$ is the state vector, and the matrices P and $Q(t)$ are positive definite. The problem is scalarized as the minimization of the weighted sum $\alpha J_1 + \beta J_2$ with $\alpha > 0$, $\beta > 0$.

The control is approximated by a piecewise-constant function on a uniform grid with m nodes. Substituting the exact solution of the state equation into the scalarized cost functional reduces the problem to a finite-dimensional quadratic optimization problem in the vector y of control values (each component of y satisfies the given bounds) [1,2].

The Python implementation `LQPSolver` performs all matrix precomputations once, enabling rapid solution of multiple instances with different weights. Three modes for generating the Pareto front are provided:

- `sum_to_one`: $\alpha + \beta = 1$, $\alpha, \beta > 0$ (relative importance);
- `lambda`: $\alpha = 1$, β varies on a logarithmic scale;
- `free`: arbitrary grid of α and β.

As a numerical example we consider the linearized vertical dynamics of a Boeing 737-800 aircraft in turbulence. The state vector comprises altitude, airspeed and flight-path angle; the single control is the normalized engine thrust. The time horizon is $[0, 300]$ s with $m = 60$ control nodes. The initial condition corresponds to a turbulence-induced perturbation.

The constructed Pareto fronts clearly illustrate the trade-off between state recovery quality and control effort. When state penalization dominates, the control exhibits sharp impulses (up to 15 % thrust deviation). When control penalization dominates, the optimal strategy is to apply zero additional thrust, leaving the aircraft in the perturbed state.

The proposed method provides an efficient numerical approximation of the Pareto front for convex parts of the set and can be extended to more general classes of multi-objective optimal control problems.

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On Solvability to Certain Elliptic Equation

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The paper considers an elliptic pseudodifferential equation in a tetrahedral cone of three-dimensional space. We investigate the solution to a boundary value problem with an additional integral condition as the cone parameters tend to their limit values. It is shown that, in this case, a solution exists in the Sobolev–Slobodetskii space.

Keywords: pseudo-differential equation, cone, asymptotic behavior of solution, wave factorization

Model homogeneous pseudo-differential equation under consideration is the following

$$(Au)(x) = 0, \qquad x \in C_3, \tag{1}$$

where A – pseudo-differential operator with symbol $A(\xi)$, which satisfies the following condition

$$c_1(1+|\xi|)^\alpha \le |A(\xi)| \le c_2(1+|\xi)^\alpha, \qquad \alpha \in \mathbb{R}, \quad c_1, c_2 > 0.$$

Cone C_3 is the intersection of five half-spaces:

$$x_3 > ax_1+bx_2, \quad x_3 > cx_1+bx_2, \quad x_3 > cx_1+dx_2, \quad x_3 > ax_1+dx_2, \quad x_3 > 0, \quad a,b,c,d > 0.$$

Let's add an integral condition

$$\int_{-\infty}^{+\infty} u(x)dx_3 = g(x'). \tag{2}$$

In [1] it was proved that in the presence of a wave factorization [2] of the symbol $A(\xi) = A_=(\xi)A_{\neq}(\xi)$ with respect to the cone C_3 with index æ such that the condition $æ - s = 1+\varepsilon$, $|\varepsilon| < 1/2$ is satisfied, the solution to the boundary value problem (1), (2) exists and is unique. To describe its structure, a special transformation operator V_φ is introduced [1,3]. For compactness of notation, we introduce the following operators:

$$P_k = \frac{1}{2}(I+S_k), \qquad Q_k = \frac{1}{2}(I-S_k), \quad k = 1,2,$$

where I is identity operator, S_k are one-dimensional singular integral operators of the following type

$$(S_1\tilde{u})(\xi_1,\xi_2,\xi_3) = \frac{i}{\pi}v.p.\int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta,\xi_2,\xi_3)d\eta}{\xi_1-\eta}, \quad (S_2\tilde{u})(\xi_1,\xi_2,\xi_3) = \frac{i}{\pi}v.p.\int_{-\infty}^{+\infty} \frac{\tilde{u}(\xi_1,\eta,\xi_3)d\eta}{\xi_2-\eta}.$$

The following result was obtained for the limit solutions of the boundary value problem (1), (2), when the cone C_3 is transformed into a two-dimensional cone.

Theorem 1. *Limit solutions of the boundary value problem* (1)*,* (2) *for*

$$1.\ a, c \to \infty,\ b, d = const;$$
$$2.\ a, c \to \infty,\ b = d = 0;$$
$$3.\ a, c \to \infty,\ b = 0,\ d = const,$$

exist if and only if the symbol $A(\xi)$admits wave factorization for large values of a, c *relative to the cone C_3 with index æ such that æ $- s = 1 + \varepsilon$, $|\varepsilon| < 1/2$, $g \in H^{s+1/2}(\mathbb{R}^2)$ and the function g satisfies the following condition:*

$$A_{\neq}\left(\frac{t_1 + \lambda t_2}{\lambda + 1}, \xi_2, 0\right) \tilde{g}\left(\frac{t_1 + \lambda t_2}{\lambda + 1}, \xi_2\right) = (P_1 A_{\neq} \tilde{g})(t_1, \xi_2) + (Q_1 A_{\neq} \tilde{g})(t_2, \xi_2),$$

where $t_1 = \xi_1 - a\xi_3$, $t_2 = \xi_1 + c\xi_3$, $\lim\limits_{a,c\to\infty} \dfrac{a}{c} = \lambda$. Although the condition is the same in all three cases, each case was treated as a separate one because the geometry of the cone is different for each of them.

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An Online Optimization Model for Point-of-Sale Credit Decisions with Aggregated Portfolio Constraints

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This paper addresses the problem of real-time credit decision-making in point-of-sale lending under aggregated portfolio constraints. Classical approaches either ignore portfolio dynamics or rely on offline optimization, which is infeasible in a fully online setting where future loan applications are unknown. We consider a hybrid online optimization model that combines a local profit-maximizing optimizer with an adaptive correction mechanism that respects daily volume caps, high-risk loan share limits, and regulatory constraints. The model uses streaming machine learning predictions of default and fraud probabilities. We formulate the expected profit function, define dynamic regret relative to an offline optimal benchmark, and outline a numerical evaluation plan comparing several baselines. The objective is

to achieve sublinear dynamic regret with respect to the offline optimal sequence of decisions under portfolio constraints, i.e., $\text{DynRegret}/T \to 0$ as $T \to \infty$.

Keywords: online optimization, POS lending, portfolio constraints, real-time decision making, regret analysis

Introduction

In modern point-of-sale (POS) credit systems, the decision to approve a loan and the loan amount must be made in real time, often within fractions of a second. Classical approaches using static scorecards and fixed limits ignore the dynamics of the bank's portfolio, current risk limits, and regulatory constraints. Meanwhile, standard online convex optimization methods typically minimize regret with respect to a fixed optimal solution but do not adapt to time-varying, aggregated portfolio constraints that couple decisions across time.

In this work, we formulate and investigate a computationally efficient online decision model for POS lending that:

1. uses streaming ML predictions of default probability $p_t^{\text{default}}(L)$ and fraud probability p_t^{fraud} for each request t;
2. selects a loan amount $L_t \in \mathcal{L} \subset \mathbb{R}_+$ (or rejection) to maximize expected profit $\pi_t(L)$;
3. satisfies aggregated portfolio constraints (daily volume cap, high-risk loan share, usury limits) in real time without knowledge of future requests.

Statement of Problem

Let $t = 1, \dots, T$ denote a sequence of credit applications. For each application t, we observe client features $\mathbf{x}_t$ and the requested amount L. Two ML models provide:

$$p_t^{\text{default}}(L) = \mathbb{P}(\text{default} = 1 \mid \mathbf{x}_t, L), \qquad p_t^{\text{fraud}} = \mathbb{P}(\text{fraud} = 1 \mid \mathbf{x}_t).$$

The expected profit from granting amount L is:

$$\pi_t(L) = (1 - p_t^{\text{fraud}})\Big[(1 - p_t^{\text{default}}(L))R(L) - p_t^{\text{default}}(L)\,\text{Loss}(L)\Big] - p_t^{\text{fraud}}\,\text{FraudLoss}(L),$$

where $R(L)$ is the interest income, $\text{Loss}(L)$ is the loss given default, and $\text{FraudLoss}(L)$ is the loss due to fraud. Rejection corresponds to $L = 0$ with $\pi_t(0) = 0$.

The aggregated portfolio constraints over a day (sum over t in that day) are:

$$\sum_t L_t \cdot \chi_{\{L_t > 0\}} \le C_{\text{day}}^{\max}, \quad \sum_t \chi_{\{p_t^{\text{default}}(L_t) > \theta_{\text{high}}\}} \cdot L_t \le \rho, \quad \text{and regulatory caps.}$$

We consider a hybrid online optimization model that combines:

- ❑ Local optimizer: for each request t, compute the unconstrained optimal amount $\tilde{L}_t = \arg\max_{L \in \mathcal{L}} \pi_t(L)$.
- ❑ Adaptive correction mechanism: track real-time consumption of portfolio limits. When limits are approached, apply:
  - ❑ amount scaling: $L_t = \min(\tilde{L}_t, \alpha_t \cdot \tilde{L}_t)$;
  - ❑ dynamic profit threshold: approve only if $\pi_t(\tilde{L}_t) > \text{threshold}_t$;
  - ❑ temporary blocking of high-risk applications.

- ❑ Online parameter tuning: update $\alpha_t, \text{threshold}_t$ based on a sliding window of past applications without looking into the future.

We consider a numerical evaluation on historical POS lending data comparing:

1. Baseline: maximize $\pi_t(L)$ ignoring portfolio constraints.
2. Hard-cut: reject once the daily limit is exceeded.
3. Proposed adaptive online model with parameter correction.
4. Offline optimal (with full day knowledge) – to measure regret.

Expected Results and Metrics

- ❑ Cumulative profit over the test period (days/months).
- ❑ Constraint satisfaction: fraction of days where portfolio limits are violated.
- ❑ Dynamic regret with respect to the best sequence of decisions respecting constraints (benchmark):
$$\text{DynRegret}_T = \sum_{t=1}^{T} \pi_t(L_t^{\text{opt}}) - \sum_{t=1}^{T} \pi_t(L_t),$$
where L_t^{opt} is the offline optimal decision given full day knowledge. The goal is to achieve sublinear dynamic regret.
- ❑ Decision latency (target < 100 ms per request).

The primary goal of this work is to develop an online decision model for POS lending that achieves sublinear regret with respect to the offline optimal benchmark. Formally, we require: $\text{DynRegret}_T = o(T)$, i.e., the average per-step regret vanishes as the time horizon T grows. This guarantees that the adaptive online model asymptotically performs no worse than the best fixed strategy in hindsight, while strictly respecting aggregated portfolio constraints in real time.

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Development of an Automated Stochastic Testing Methodology with Scenario Coverage Assessment and Adaptive Generation of Critical Situations

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This study addresses the development and evaluation of navigation algorithms for autonomous transport systems in environments with static obstacles and constrained operational zones. We propose a lightweight 2D simulation framework for deterministic testing of local path planners, eliminating the computational overhead of full physics engines. Two novel pathfinding algorithms are synthesized: a repulsion-based method that maximizes clearance from obstacles, and an attraction-based method that optimizes heading utility toward the goal. Experimental results demonstrate that the attraction-based approach achieves significantly higher success rates (85–95 %) across different map sizes and obstacle densities, while the repulsion-based method suffers from reliability issues in complex environments. The findings confirm the effectiveness of simplified testing frameworks for algorithmic evaluation and provide practical recommendations for industrial deployment.

Keywords: pathfinding algorithms, autonomous transport systems, simulation model

Introduction

The proliferation of autonomous transport systems demands reliable local trajectory planning algorithms capable of operating in environments with static obstacles and constrained operational zones. The objective of this study is to review existing navigation methods, synthesize novel algorithms and supporting software, and provide an objective evaluation of their effectiveness using a specialized simulation model. To achieve this, we focus on developing pathfinding algorithm software, generating test scenarios with varying complexity, and conducting a comparative analysis of methods across different vehicle platforms.

Existing navigation approaches include several key methods. The cell-based method discretizes the workspace into a grid and is effective for pre-mapped static environments, with accuracy improving with map resolution at the cost of increased computational load. The geometric method constructs a collision threat zone and assumes a maneuver that maximizes lateral clearance, but performs reliably only for isolated or low-mobility obstacles. The potential field method relies on vector superposition of attractive (goal-directed) and repulsive (obstacle-avoidance) forces, but is susceptible to local minima in the presence of non-convex obstacles. Finally, randomized exploration trees (RRT-like) iteratively construct a path graph in configuration space, yet require a priori knowledge of the obstacle map to verify node connectivity.

Developed Models and Test Bench

Our environment model represents a 2D discrete plane as a binary matrix, where 0 denotes free space and 1 indicates obstacles or boundaries. The perception system emulates a scanning LiDAR, outputting a vector of range measurements within a specified angular sector relative to the vehicle heading. To isolate algorithmic logic from sensor noise and physical dynamics for focused evaluation, we introduce several model simplifications: perfect localization, instantaneous scanning, static obstacles within a single iteration, and absence of dynamic constraints on turning radius. We justify avoiding heavy simulators like Gazebo and AirSim by their high per-iteration computational cost, complexity of automatic parameterization, and redundancy for heuristic comparison tasks. Instead, we propose a lightweight deterministic 2D test bench. Location generation involves automatic random placement of convex and non-convex objects with intersection control and adjustable occupancy density, with reproducibility ensured via a fixed random seed.

Synthesized Pathfinding Algorithms

We have synthesized two pathfinding algorithms. The repulsion-based method selects the heading that maximizes clearance from the current position while progressing toward the goal, based on identifying local maxima in the range vector. The attraction-based method evaluates the "utility" of a heading by the degree of approach to the target, accounting for sufficient clearance for subsequent maneuvering. The control signal in this method is derived from maximizing this utility metric.

Experimental Evaluation Results

Experimental evaluation included two main tests. In Test 1, conducted on a 20×20 map with occupancy $\approx$ 11,25%, the repulsion method achieved a success rate of $\approx$ 49−56% with an average flight time of $\approx$ 271 time units, while the attraction method demonstrated a success rate of $\approx$ 92−95% with an average flight time of $\approx$ 275 time units. Test 2, performed on a 50×50 map with occupancy $\approx$ 18%, revealed a sharp reliability drop for the repulsion method ($\approx$ 13−15% success rate, time $\approx$ 960–980 time units), whereas the attraction method maintained high efficiency ($\approx$ 85−87% success rate, time $\approx$ 723–725 time units). Analysis of vehicle type influence showed minimal differences between airplane-type and helicopter/quadcopter-type platforms within the model, confirming algorithmic robustness to kinematic variations.

Conclusion

The repulsion-based method proves critically sensitive to obstacle geometry: non-convex shapes are frequently misinterpreted as traversable passages, leading to erroneous routes and entrapment. In contrast, the attraction-based method demonstrates substantially higher reliability and capability to solve more complex tasks, generating more optimal trajectories. Both algorithms guarantee task completion primarily in environments devoid of non-convex obstacles; in their presence, success cannot be guaranteed, but the attraction-based method remains significantly more robust. The developed lightweight 2D test bench enables large-scale deterministic testing of local planners without the computational overhead of full physics engines. For industrial deployment, integration of the synthesized methods with global planning modules or augmentation with heuristics for local-minimum escape is recommended.

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